

Universal Scaling in Complex Substitutive Systems

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Diffusion processes are central to human interactions. Despite extensive studies that span multiple disciplines, our knowledge is limited to spreading processes in non-substitutive systems. Yet, a considerable number of ideas, products and behaviors spread by substitution—to adopt a new one, agents must give up an existing one. Here, we find that, ranging from mobile handsets to automobiles to smart phone apps, early growth patterns in substitutive systems follow a power law with non-integer exponents, in sharp contrast to the exponential growth customary in spreading phenomena. Tracing 3.6M individuals substituting for mobile handsets for over a decade, we uncover three generic ingredients governing substitutive processes, allowing us to develop a minimal substitution model, which not only predict analytically the observed growth patterns, but also collapse growth trajectories of constituents from

rather diverse systems into a single universal curve. These results demonstrate that the dynamics of complex substitutive systems are governed by robust self-organizing principles that go beyond the particulars of individual systems, which implies that these results could guide the understanding and prediction of all spreading phenomena driven by substitutions, from electric cars to scientific paradigms, from renewable energy to new healthy habits.

Diffusion processes impact broad aspects of human society (1–5), ranging from the spread of biological viruses (3, 6–8) to the adoption of innovations (4, 9–13) and knowledge (14, 15) and to the spread of information (16, 17), cultural norms and social behavior (18–21). Despite the extensive studies that span multiple disciplines, our knowledge is limited to spreading processes in non-substitutive systems. Yet, a considerable number of ideas, products and behaviors spread by substitution—to adopt a new one, agents must give up an existing one (Fig. 1). Indeed, just as creationism and evolutionism rarely coexist in individual beliefs, Kuhn’s notion of paradigm shifts highlights the fact that the development of science hinges on scientists’ relentlessness in abandoning a scientific framework once one that offers a better description of reality emerges (22). The same is true for adopting a new healthy habit or other durable items, like mobile phones, cars or homes. For example, the development of health behavior can be only viewed as a substitutive process—one must first break away from the old habit to adopt the new.

While substitutions play a key role from science to economy, our limited understanding of such processes stems from the lack of empirical data tracing their characteristics. To study the dynamics of substitution processes, we explore growth patterns in three different complex substitutive systems where detailed spreading process is captured with fine temporal resolution (See supplementary materials S1 for detailed data descriptions). *D1*: A decade-long mobile phone dataset that captures, with daily resolution, the choice of 3.6 Million individuals among 8,928 different types of mobile handsets for over a decade, recorded by a Northern European telecommunication company from January 2006 to November 2014. Since an individual is unlikely to keep more than one mobile phone at a time, his or her adoption of a new handset is associated with discontinuance of the old one. For each handset, we calculated the number of individuals $I(t)$ who bought the handset up to time t since its availability (Fig. 2A). *D2*: An automobile dataset captures monthly transaction records of all automobiles sold in the North America (126 different car models between 2010 and 2016). Similar to mobile handsets, auto-

mobiles represent another exemplary case where substitutions drive adoptions, given the limited number of automobiles a typical household may have. Here the impact of each automobile is captured by its cumulative sales over time (Fig. 2D). Finally, using $D3$, we traced new smart phone apps published in the App store (2,672 most popular apps from November to December 2016), counting the number of daily downloads for each app (Fig. 2G). Similarly, usages of smart phone apps are subject to constraints of time and device space, hence a new app downloaded reduces the usage of other similar apps, if not replacing them all together. To avoid potential selection bias towards highly popular apps, we also explored a more uniform sample, containing all new applications belonging to the Health and Fitness category (supplementary materials S1.1, fig. S2).

The three systems differ widely in their scopes, scale, temporal resolution and user demographics. We find, however, that independent of the nature of the system and the identify of the constituents, the early growth patterns across all of them are characterized by power laws with varying exponents (Fig. 2). For example, we find that the number of users for each new handset shown in Fig. 2A grows as

$$I(t)/I(1) = t^{\eta_i}. \quad (1)$$

To compare different handsets, we normalized $I(t)$ by the number of users on the first day of release $I(1)$. While different handsets all follow power law growth curves, their slopes differ, each characterized by handset-specific exponents (η_i). Hence if we plot each curve on Fig. 2A in terms of t^{η_i} , all handsets collapse onto the same universal line $y = x$ (Fig. 2B).

We also find that, exponents η_i are mostly non-integers (Fig. 2C). Power law growth with such non-integer exponents is rather striking, due to the inability to express them in terms of Taylor series around $t = 0$, corresponding to non-analytic behavior around the release time. Indeed, current modeling frameworks, from epidemiological models (3,6) to disordered systems (2) to diffusion of innovations (4,11), predict an exponential early growth pattern, and are unable

to anticipate power laws with non-integer exponents (Detailed descriptions and comparisons of existing models are described in supplementary materials S3).

We repeated our analyses for automobiles (*D2*, Fig. 2D–F) and smartphone apps (*D3*, Fig. 2G–I), uncovering the same universal power law scaling spanning large orders of magnitude in time. Given the myriad factors that determine the dynamics of spreading processes, ranging from initial seeds (*I2*, *20*), timing and social influence (*23*, *24*) to a large set of often unobservable factors (*25*), the universal scaling documented in Fig. 2 is rather unexpected, highlighting a radical departure from our current understanding of diffusion processes. Indeed, comparing with exponential growth, power law encodes an early divergence, corresponding to an explosive growth at the moment when new substitutes are introduced. Yet following this brief singularity, the number of users grows much more slowly than what exponential functions predict, suggesting that substitutive innovations spread more slowly beyond the initial excitement. To be sure, power laws can be generated in real networks due to the growth of the systems (*26*, *27*). To check if the power laws observed in Fig. 2 may be explained by gradual addition of new users to the underlying network, we removed new mobile subscribers in *DI* and measured again $I(t)$ for different handsets. We find that the power law scaling remains unchanged (supplementary materials S3.4, fig. S8), indicating that the scaling observed in Fig. 2 is governed by fundamentally different mechanisms that operate *within* the system, not driven by growth of the system. Together, these results raise a critical question: what is the origin of the power law growth pattern?

A common characteristic of the three studied systems is that they evolve by substitutions. In this respect, mobile phones represent among the most ideal settings for the empirical investigations of such complex substitutive systems. Indeed, each time a user purchases a new handset, the transaction history is recorded by telecommunication companies. Anonymized phone numbers together with their portability across devices provide excellent individual traces

for substitutions. In other words, the mobile phone dataset (*DI*) may offer us a unique opportunity to quantify substitution patterns. To this end, we examined detailed user histories in *DI*, finding that the adoption and discontinuance histories are indeed predominantly represented by substitutions (supplementary materials S2.2). Each type of handsets is substituted by a large number of other handsets, hence substitution patterns are characterized by a dense, heterogeneous network that evolves rapidly over time ($\langle\langle k \rangle\rangle \approx 73.6$, figs. S6BC & S7). To visualize substitution patterns, we applied a backbone extraction method (28) to identify statistically significant substitution flows for each handset given its total substitution volumes (Fig. 3A). While mobile handsets have changed substantially over the course of ten years, undergoing a ubiquitous shift from feature phones to smart phones, the rate at which new handsets enter the market remained remarkably stable (Fig. 3B), highlighting the highly competitive nature of the system. Indeed, ensuing generations of new handsets enter the market in a somewhat regular manner, substituting for the incumbent, thereby driving the rise and fall in their popularities (fig. S7A).

To uncover the mechanisms governing the dynamics of substitutions, we note that the rate of change in N_i , the number of users for handset i , can be expressed in terms of the probability for individuals to transition from all other handsets (k) to i , $\Pi_{k \rightarrow i}$, subtracted by those leaving i for other handsets (j), $\Pi_{i \rightarrow j}$:

$$\frac{dN_i}{dt} = \sum_k \Pi_{k \rightarrow i}(t) N_k - \sum_j \Pi_{i \rightarrow j}(t) N_i. \quad (2)$$

Hence the key to solving the master equation (2) is to determine $\Pi_{i \rightarrow j}$, the substitution probability for an user to substitute handset i for j at time t . As we show next, $\Pi_{i \rightarrow j}$ is driven by three fundamental mechanisms: *preferential attachment*, *recency* and *propensity*.

Figure 3C shows that $\Pi_{i \rightarrow j}$ is independent of the number of individuals using i (N_i), but proportional to those using j (N_j). This result indicates $\Pi_{i \rightarrow j} \sim N_j$, which is consistent with existing models that can be used to characterize substitutions (29–31), documenting the well-

known preferential attachment effects (14, 26): users are more likely to substitute handset i with j if j is more popular than other available handsets. Yet N_j by itself is insufficient to explain $\Pi_{i \rightarrow j}$. Indeed, we normalized $\Pi_{i \rightarrow j}$ by N_j , by defining $S_{i \rightarrow j} \equiv \Pi_{i \rightarrow j}/N_j$, the substitution rate at which handset j substitutes for i . We find that $P(S_{i \rightarrow j})$ follows a fat-tailed distribution spanning several orders of magnitude (Fig. 3D). Hence, after accounting for the popularity of substitutes, substitution rates are yet characterized by a high degree of heterogeneity, where $S_{i \rightarrow j}$ between some handset pairs are orders of magnitude higher than others.

Identifying mechanisms responsible for the observed heterogeneity in $S_{i \rightarrow j}$ leads us to uncover two new mechanisms governing substitutions. Indeed, we grouped $S_{i \rightarrow j}$ based on the age of the substitutes t_j , the number of days elapsed since its release date, and measure the conditional probability $P(S_{i \rightarrow j}|t_j)$ for each corresponding group. We find that as substitutes grow older (increasing t_j), $P(S_{i \rightarrow j}|t_j)$ shifts systematically to the left (Fig. 3E), indicating substitution rates decrease with the age of substitutes—newer handsets substitute for the incumbents at a higher rate. Yet, within each group, $P(S_{i \rightarrow j}|t_j)$ again follows a fat-tailed distribution, indicating the heterogeneity of $S_{i \rightarrow j}$ persists even for handsets with the same degree of freshness. Once we rescale the distributions $P(S_{i \rightarrow j}|t_j)$ with t_j , however, all seven distributions in Fig. 3E collapse into one single curve (Fig. 3F), demonstrating that a single universal distribution characterizes substitution rates, independent of the age of substitutes:

$$P(S_{i \rightarrow j}|t_j) \sim t_j \mathcal{F}(S_{i \rightarrow j} t_j). \quad (3)$$

In other words, substitution rates $S_{i \rightarrow j}$ can be decomposed into two independent factors: one is the universal function $\mathcal{F}(x)$, which is independent of the substitute's age, capturing an inherent propensity-based heterogeneity among handsets. Denoting the propensity by $\lambda_{ij} \equiv S_{i \rightarrow j} t_j$, (3) indicates $S_{i \rightarrow j} \sim \lambda_{ij} \frac{1}{t_j}$. Repeating our analysis for t_i , i.e., the age of incumbent handset i when substituted, we find eight curves of $P(S_{i \rightarrow j}|t_i)$ automatically collapse onto each other (Fig. 3G).

Hence, when incumbents are substituted, whether they were released merely a few months ago (small t_i) or have existed in the market for years (large t_i), their substitution rates follow the same universal distribution, documenting a surprising independence between substitution rates and the age of the incumbents, in sharp contrast to the systematic temporal shifts observed in $P(S_{i \rightarrow j}|t_j)$. Mathematically, Fig. 3G indicates $P(S_{i \rightarrow j}|t_i) = P(S_{i \rightarrow j})$.

Together, Figs. 3E–G help us uncover two novel mechanisms governing substitutions: *recency* and *propensity*: substitution rates depend on the recency of substitutes, following a power law $1/t_j$. The uncovered power law decay has a simple origin, documenting the role of competitions in driving the obsolescence of handsets. Indeed, when j first entered the system, being the latest handset (small t_j), it substitutes for the incumbent at its highest rate. Yet with time, more and more newer handsets are introduced. The constant arrivals of new handsets (Fig. 3B) imply that the number of alternatives to j grows linearly with t_j . Hence if we pick one handset randomly, the probability for handset j to stand out among its competitors decays as $1/t_j$. The temporal decay is further modulated by the inherent propensity λ_{ij} between two handsets, capturing the extent to which a certain handset is more likely to substitute for some handsets than others. Mathematically, Figs. 3C–G predict

$$\Pi_{i \rightarrow j} = \lambda_{ij} N_j \frac{1}{t_j}. \quad (4)$$

Most importantly, (4) defines a Minimal Substitution (*MS*) model, which, as we show below, naturally leads to the observed power law growth phenomena. In this model, the system consists of a fixed number of individuals, with new handsets being introduced constantly. In each time step, each individual user substitutes his or her current handset i for new handset j with probability $\Pi_{i \rightarrow j}$, according to (4). The propensity λ_{ij} between handset i and j is determined by drawing a random number from a fixed distribution. We show that the results are independent of specific distributions λ_{ij} follows (supplementary materials S4.1–4.3).

Plugging (4) into (2) enables us to predict the dynamics of substitutions analytically. Indeed, the first term in the right hand side of (2) determines the impact of handset i , i.e., its cumulative sales:

$$I_i(t_i) = \int_0^{t_i} \sum_k \Pi_{k \rightarrow i} N_k dt. \quad (5)$$

Solving the master equation (2) analytically yields (supplementary materials S4.4):

$$\begin{aligned} N_i(t_i) &= h_i t_i^{\eta_i} e^{-t_i/\tau_i} \\ I_i(t_i) &= h_i \eta_i \tau_i^{\eta_i} \gamma_{\eta_i}(t_i/\tau_i), \end{aligned} \quad (6)$$

where $\gamma_{\eta}(t) \equiv \int_0^t x^{\eta-1} e^{-x} dx$ is the lower incomplete gamma function. Equation (6) indicates that the number of individuals using handset i at any moment t_i is determined by three parameters η_i , h_i and τ_i : $\eta_i \equiv \sum_k \lambda_{k \rightarrow i} N_k$ captures the *fitness* of a handset, measuring the total propensity for users to switch from all other handsets to i . The *anticipation* parameter h arises from the boundary condition at $t_i = 0$ when solving the differential equation (2). According to (6), h is equivalent to the number of individuals using handset i when $t_i = 1$, capturing the initial excitement of users for a particular handset. τ_i is the *longevity* parameter, as it captures the characteristic time scale for i to become obsolete in face of ensuing new handsets. Indeed, defining t_i^* as the time when a handset reaches its maximum number of users N_i^* , (6) predicts that the peak time t_i^* is proportional to the longevity parameter and fitness: $t_i^* = \eta_i \tau_i$.

In the early stage of a lifecycle (small t_i), (6) predicts that the impact of handset i grows following a power law:

$$I_i(t_i) = h_i t_i^{\eta_i}, \quad (7)$$

where the growth exponent is uniquely determined by the fitness parameter η_i , governed by its propensity to substitute for the incumbents in the system. The higher the fitness, the faster is the take-off in the number of users, which is modulated by the anticipation parameter h , capturing the impact difference on the release date of a handset.

The *MS* model not only explains the early growth phase. It also predicts accurately the entire lifecycle of impacts (supplementary materials S4.6). By using the rescaled variables: $\tilde{t}_i = t_i/\tau_i$ and $\tilde{I}_i = I_i/(h_i\eta_i\tau_i^{\eta_i})$, we obtain:

$$\tilde{I}_i = \gamma_{\eta_i}(\tilde{t}_i). \quad (8)$$

Therefore, for handsets with the same fitness, their impact dynamics can be collapsed into the same universal function after rescaling by the three independent parameters in the model (η , τ and h). Most strikingly, since the rescaling formula (8) we derived is independent of the particulars of a system, it predicts that, constituents from *different* systems should all follow the *same* universal curve as long as they have the same fitness. That is, not only different handsets with the same η should collapse into the universal formula (8), impact dynamics of automobiles or mobile apps that have the same η should also be captured by the same universal curve.

To test these predictions, we fit our model (6) to all three systems using maximum-likelihood estimation (supplementary materials S4.5) to obtain the best-fitted three parameters (η_i , h_i , τ_i) for each handsets, automobiles, and apps. We first selected from the three systems, for those that have similar fitness ($\eta \approx 1.5$). Although the impact dynamics appear widely different from each other (Fig. 4A–C), we find all curves simultaneously collapse into one single curve after rescaling using (8) (Fig. 4D–F). To test the universal formula (8) for variable fitness, we select two additional groups of handsets ($\eta \approx 1.8$ and $\eta \approx 2.0$), finding that the rescaled impact dynamic in both groups well collapse into their respective universality classes predicted by (8) (Fig. 4G–H). The universal curves correspond to the associated classes of the incomplete gamma functions $\gamma_{\eta_i}(\tilde{t}_i)$, which only depend on the fitness parameter η (Fig. 4I). The model also predicts that if we properly normalize out the effect by $\gamma_{\eta_i}(\tilde{t}_i)$, we can rescale the entire lifecycle to a power law solely governed by η . Indeed, (6) indicates that, by defining $Q \equiv [I(t)/h - \tau^\eta \gamma_{\eta+1}(t/\tau)] e^{t/\tau}$, Q grows following a power law, $Q = t^\eta$. We find excellent agreement in all three systems we studied (Figs. 4J–L), further corroborating our modeling framework. Together Figs. 4A–L doc-

ument remarkably universal impact dynamics predicted by our model that not only hold *within* a system but *across* different complex substitutive systems. Given the obvious diversity of the studied systems and constituents within them, this level of universality is truly unexpected.

The *MS* model predicts an intriguing underlying connection between short-term and long-term impact. Indeed, we can calculate the ultimate impact—the total number of a particular handset, automobile, or mobile app, ever sold or downloaded in its lifetime—by taking the $t \rightarrow \infty$ limit in (6), obtaining:

$$I_i^\infty = h_i \Gamma(\eta_i + 1) \tau_i^{\eta_i}. \quad (9)$$

Comparing (6) and (9) reveals that ultimate impact and the impact when the number of users was at its peak follow a simple scaling relationship

$$\frac{I_i^\infty}{I_i(t_i^*)} = \Phi(\eta_i), \quad (10)$$

where $\Phi(\eta) \equiv \frac{\Gamma(\eta)}{\gamma_\eta(\eta)}$. That is, I_i^∞ scales linearly with peak impact $I_i(t_i^*)$, and the ratio between the two is determined only by the initial power law growth exponent η_i . To validate (10) we find I_i^∞ and $I_i(t_i^*)$ follow a clear linear relationship in our dataset for different values of η (Fig. 4M). In addition, Fig. 4M shows the relationship posts a slight shift as η increases. The rather subtle shift is accurately predicted by (10), as $\Phi(\eta)$ increases slowly with η (Fig. 4N). Therefore, the uncovered power law growth patterns offer an explicit link between short-term and long-term impact in substitutive systems, providing a simple yet intriguing formula that allows us to estimate the ultimate impact of any constituent once its number of users reaches its peak (supplementary materials S4.7).

In summary, here we collected a diverse set of large-scale data pertaining to substitutive processes, finding that ranging from mobile handsets to automobiles to smart phone apps, early growth patterns in substitutive systems do not follow the exponential growth customary in spreading phenomena. Instead, substitution processes grow following power law with non-

integer exponents, indicating that they start with an initial explosive adoption process, followed by a much slower growth than expected in normal diffusion. Analyzing 3.6 Million individuals substituting for 8,928 different types of mobile handsets for everyday within a ten-year period, we uncovered three generic ingredients governing substitutions. Incorporating these ingredients, we develop a minimal model for substitutions, allowing us to not only predict analytically the power law growth patterns observed in real substitutive systems, but also to collapse growth trajectories of constituents from rather diverse systems into a single universal curve.

Together, the results reported in this paper not only unpack the origin of robust self-organization that emerged in complex substitutive systems, but also demonstrate a high degree of universality across such systems. Given the ubiquitous role substitutions play in a wide range of important settings, our results may have applicabilities reaching beyond the quoted examples. Indeed, these results could guide the understanding and prediction of all spreading phenomena driven by substitutions, from electric cars to scientific paradigms, and from renewable energy to new healthy habits, opening up a rather promising direction for future investigation. Lastly, to what degree can our results be extended beyond substitutive systems is a question we cannot yet answer conclusively. Sub-exponential growth patterns are recently observed in spreading of certain diseases like Ebola and HIV (32, 33), which suggest power law growth explained in our work may exist in even broader domains.

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Supplementary Materials.

Materials and Methods

Supplementary Text

Figs. S1 to S12

Table. S1

References (34-85)

Figure 1. Illustrative Comparison between Epidemic Processes and Substitutive Processes. In an *epidemic process* (Left panel), agents adopt new items as an independent process. In epidemiological models for example, adoption of new items is modeled as transitions from a susceptible to infectious state. The early growth in the number of infectious users follows a multiplicative process, best approximated by an exponential function (3, 6). In a *substitutive process* (Right panel), agents can only take one or a small number of items at a time. As a result, for one to adopt a new item j , the agent must first give up the incumbent item i . The early growth pattern in such substitutive systems is found to follow power laws with non-integer exponents. Without loss of generality, we denote with i the incumbent and j the substituting item in this paper.

Figure 2. Power law growth patterns in substitutive systems. (A) Normalized impacts of 240 different handsets as a function of time. We use the first six months to measure the early growth phase for each handset by focusing on handsets that have been released for at least six months (supplementary materials S1.1, S1.2). We find, for a substantial fraction of handsets (240 handsets out of 885, 27.12%), their early growth patterns can be well approximated by power laws ($R^2 > 0.99$): $I(t) \sim t^\eta$. To compare different curves, we normalized $I(t)$ by $I(1)$, the number of users on the first day of release (SM S1.3). The color of the line corresponds to the associated power law exponent for each handset, η . The solid lines are $y = x^{1/2}$, $y = x$, and $y = x^2$, respectively, as guides to the eye. The dashed line corresponds to an exponential function following $y \sim e^x$, highlighting its fundamentally different nature comparing with power law growth patterns. (B) We rescale the impact dynamics plotted in (A) by t^η , finding all curves collapse into $y = x$. (C) Distribution of power law exponents $P(\eta)$ for curves shown in (A). (D to F) Similar power law growth patterns are observed for automobile dataset. We use the first four months to measure the early growth phase (supplementary materials S1.2). We find, for a substantial fraction of automobiles (37 out of 126 cars, 29.37%), their early

growth patterns can be well approximated by power laws ($R^2 > 0.99$). See also supplementary materials S1.3 for more details. **(G to I)** Similar power law growth pattern is observed for smart phone apps. We show 1,022 out of 2,672 apps in total (38.25%) in (G). Given the short lifespan of mobile apps, the early growth phase is measured as the first seven days following an app's introduction to the App store (supplementary materials S1.2).

Figure 3. Empirical substitution patterns. **(A)** We used the backbone extraction method (28) to construct a substitution network, capturing substitution patterns among handsets aggregated within a six-month period (January 2014—June 2014). Each node corresponds to one type of handset released prior to 2014 by one of the six major manufacturers. Node size captures its popularity, measured by the number of users of the particular handset at the time. Handsets are colored based on their manufacturers (node coloring), which fade with the age of handsets. If users substituted handset i with j , we add a weighted arrow pointing from i to j . The link weight captures the total substitution volumes between two handsets within the six-month period. Since the full network is too dense to visualize, here we only show the statistically significant links as identified by the method proposed in Ref. (28) for p-value 0.05. We color the links based on the color of the substituting handset. The network vividly captures the widespread transitions from feature handsets to smart phones. Indeed, most cross-manufacturer substitution links are either yellow or green, indicating their substitutions by iPhones or Android handsets. Substitution patterns are also highly heterogeneous. A few pairs of handsets have high substitution volumes, e.g. between the successive generations of iPhones, but most substitutions are characterized by rather limited volumes. The structural complexity shown in (A) is further coupled with a high degree of temporal variability. Indeed, the system turns into a widely different configuration every year, even for the most dominant handsets (fig. S7B–E). **(B)** The number of new handsets launched per quarter as a function of time. We find new handsets are introduced at a constant rate. The most popular handset within each eight-month time window is highlighted by an im-

age of the handset model. **(C)** Substitution probability $\Pi_{i \rightarrow j}$ is proportional to N_j , consistent with the preferential attachment effect. Inset shows $\Pi_{i \rightarrow j}$ is largely independent of N_i . **(D)** Distribution of substitution rates $S_{i \rightarrow j}$. Here we measured the substitution rates among handsets in January 2014. **(E)** Distribution of substitution rates $S_{i \rightarrow j}$ conditional on age of the substitute t_j . The distributions shift systematically to the left as t_j increases. **(F)** After rescaling substitution rates by t_j , we measure $p(S_{i \rightarrow j} t_j | t_j)$ and find all seven curves in (E) collapse into one single curve. **(G)** Distribution of substitution rates conditional on age of the incumbent (t_i). All curves collapse automatically onto one single distribution, indicating a surprising independence between substitution rates and the age of incumbent handsets.

Figure 4. Universal impact dynamics. **(A to C)** Impact dynamics for products with similar fitness ($\eta = 1.5 \pm 0.1$), including 40 handsets (A), 9 automobiles (B) and 43 apps (C). **(D to F)** Data collapse for products shown in (A—C). After rescaling time and impact independently by $\tilde{t}_i = t_i/\tau_i$ and $\tilde{I}_i = I_i/(h_i \eta_i \tau_i^{\eta_i})$, we find all curves from three systems collapse into the same universal curve, as predicted by (8). **(G)** Data collapse for handsets with similar fitness $\eta = 1.8 \pm 0.1$ (30 handsets). **(H)** Data collapse for handsets with similar fitness $\eta = 2.0 \pm 0.1$ (22 handsets). **(I)** The universal functions shown in (D—I) are each associated with their respective universality classes that are solely determined by η . Here we visualize the analytical function $\tilde{I} = \gamma_\eta(\tilde{t})$, with $\eta = 1.5, 1.8$ and 2.0 . **(J to L)** The entire lifecycle can be rescaled as power laws if we properly normalize out the effect from the incomplete gamma functions. Indeed, because $\gamma_\eta(x)$ follows recurrence relationship $\gamma_{\eta+1}(x) = \eta \gamma_\eta(x) - x^\eta e^{-x}$, (6) predicts that by defining $Q \equiv (I(t)/h - \tau^\eta \gamma_{\eta+1}(t/\tau)) e^{t/\tau}$, we obtain $Q = t^\eta$. Here we plot Q as a function of t^η for selected products ($R^2 > 0.95$) in all three systems including 448 handsets (J), 75 automobiles (K), 1212 apps (L), finding all curves are collapsed onto $y = x$. **(M)** I^∞ as a function of $I(t^*)$ for 546 handsets with different fitness η . We fit the impact dynamic of each handset in the system by (2), and select the handsets with $R^2 > 0.9$. I^∞ and t^* are calculated

through the system parameters: h , η , and τ . $I(t^*)$ is the handset's impact at time t^* obtained from the empirical data (supplementary materials S4.7). **(N)** Scatter plot for the ratio $I^\infty/I(t^*)$ as a function of η for the same handsets shown in **(M)**. The error bar indicates one standard deviation. The solid line corresponds to the analytical prediction by (10).

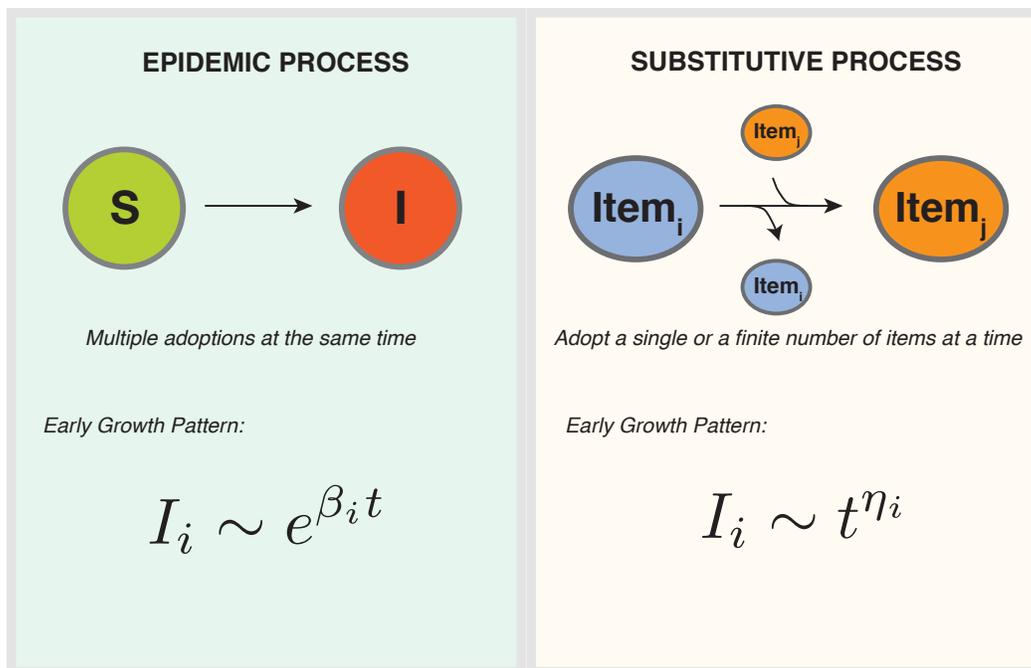


Figure 1

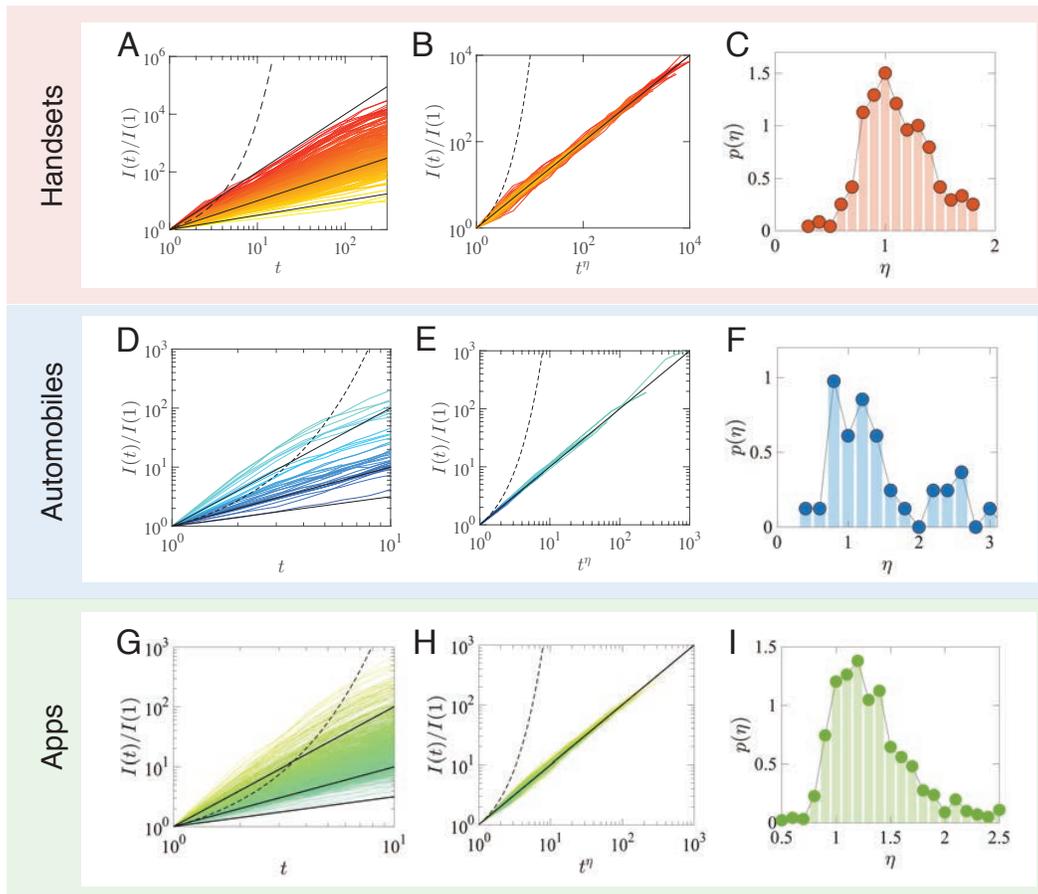
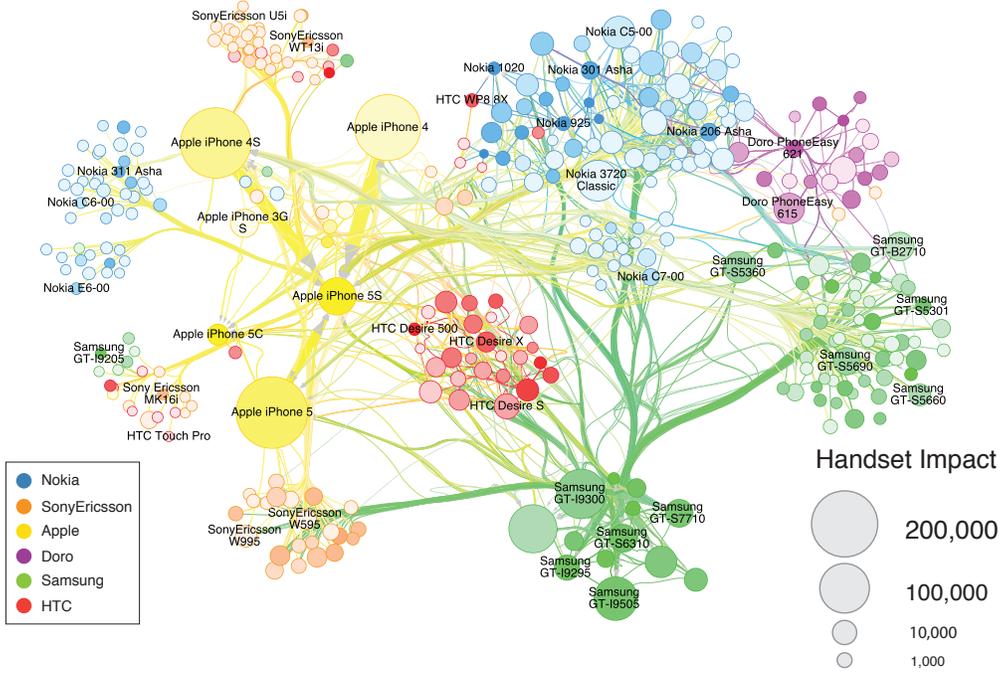
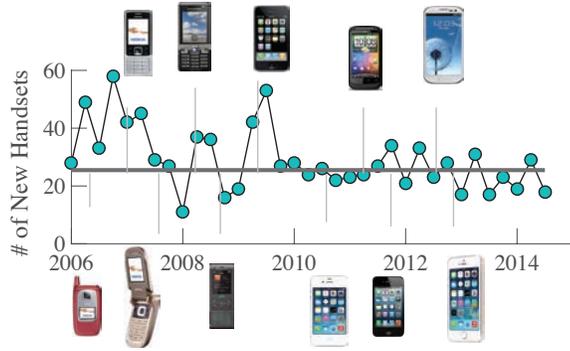


Figure 2

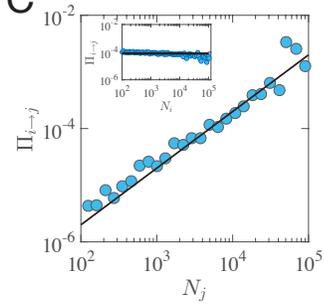
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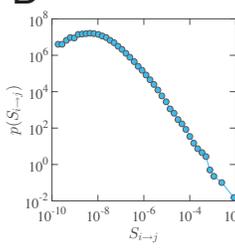
B



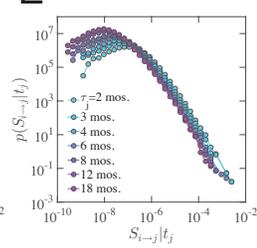
C



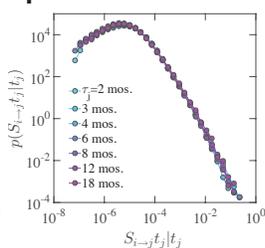
D



E



F



G

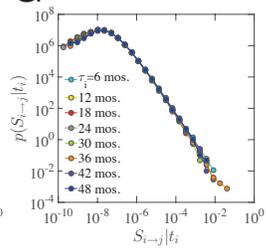


Figure 3

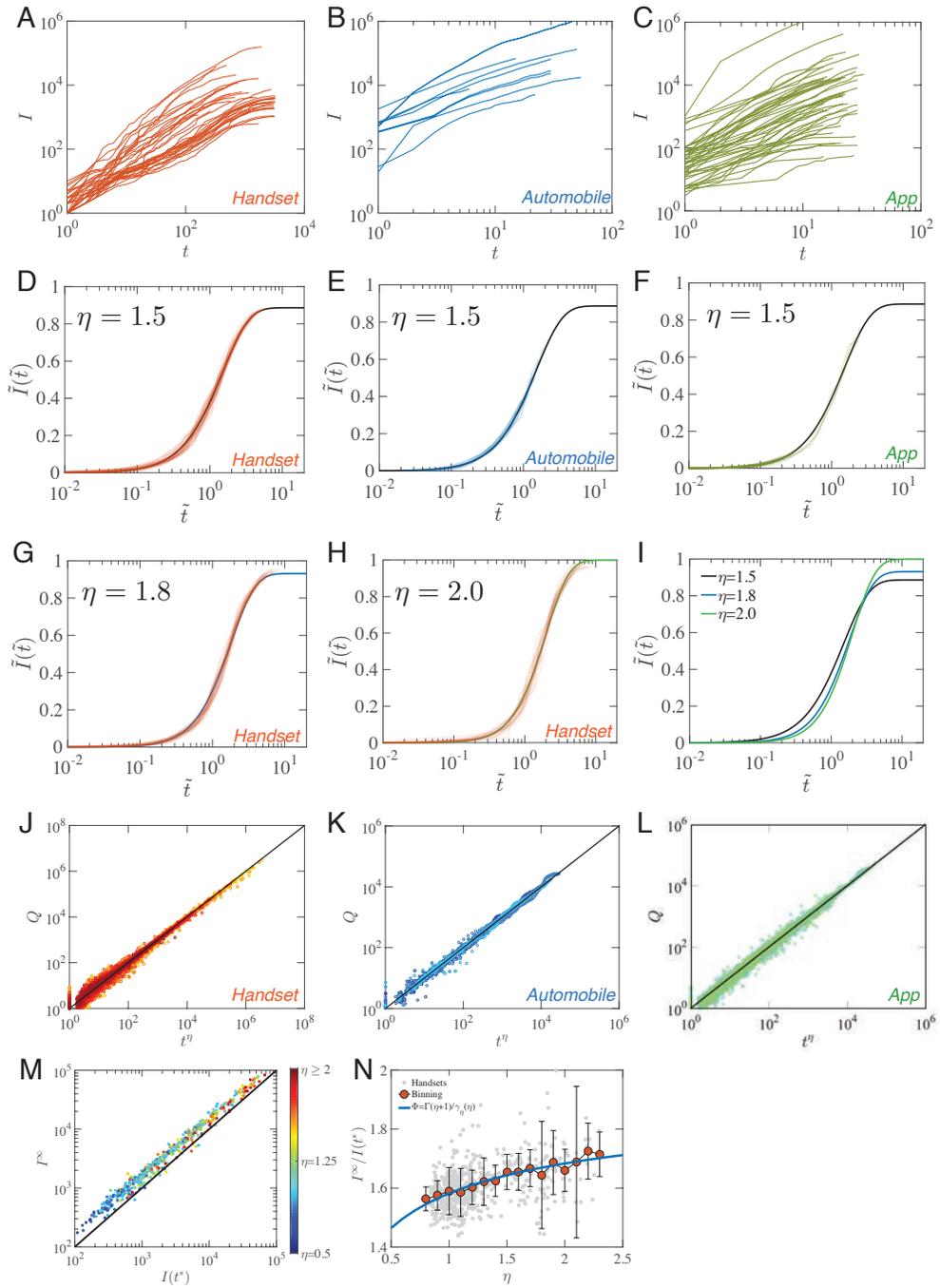


Figure 4

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Supplementary Materials for
Universal Scaling in Complex Substitutive Systems

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S1 Dataset Descriptions

S1.1 Dataset Introduction

To study early growth patterns in complex substitutive systems, we explore three different domains where large-scale databases with fine temporal resolution are available:

D1 is a mobile phone dataset recorded by a major Northern European telecommunication company. It captures daily usage patterns of 3.6 Million individuals substituting 8,928 types of mobile handsets from 01/01/2006 to 11/03/2014. By identifying each anonymized SIM-ID as an individual user, detecting the first and the last date when the individual used a particular handset, we construct the usage timeline for every handset model. To measure the impact of each handset $I(t)$, we calculated the number of individuals who bought the handset up to time t since its availability. Here, we specifically focus on 885 handset models in the dataset to study the early growth pattern of handset impacts. These handset models were chosen because they have been released for at least 180 days and have at least 50 users in total to make sure we have enough statistics for our data analysis and to be able to observe the early growth period (Definition of the early growth phase, see Sec. S1.2 and Fig. S1).

D2 is an automobile dataset collected from a website that records automobile sales data (*Good Car Bad Car*: <http://www.goodcarbadcar.net/>). The dataset captures monthly transaction records of 135 different models of automobiles sold in the U.S. and Canada from year 2010 to 2016. We focused on 126 models that have been introduced to market for more than 4 months to guarantee we have enough data to study their early growth patterns. In this dataset, we define the impact of automobiles as their cumulative sales across North America.

D3, a smartphone application dataset, captures daily-download records for the top 2,672 mobile apps released between 11/20/2016 and 12/20/2016 in the App store by Apple. These apps have been introduced for at least two weeks, allowing us to study their early growth patterns. In

this dataset, the impact of each app is defined as the cumulative number of downloads. The data are collected from a mobile application platform *Apptopia*: <https://www.apptopia.com/>. The website collects information for each mobile app and categorizes them based on their functions. For each category, the website also ranks the apps by their performance, and selects for the top apps released within the latest month.

D3s. Health and Fitness Apps. To avoid possible selection bias towards highly popular apps that may skew our empirical observations, we compiled another complementary dataset to form a more uniform sample, consisting of all 70,377 iTunes applications belonging to the category *Health and fitness* until 12/15/2016 from *Apptopia*. Since the information is most complete within 3 months, we studied 22,982 apps released after 9/15/2016. We repeated the same analysis as Fig. 2 on this dataset, uncovering same power law growth patterns (Fig. S2).

S1.2 Identifying the Early Stage of the Growth Curve

To focus on early growth patterns, we systematically define the concept *early growth phase* in three studied datasets to make sure that we are consistent across our analysis. Specifically, for each item in a given system, we identify the time T_s when the growth rate dI/dt reaches its peak ($d^2I/dt^2 = 0$). For each dataset, we define T^* as the position of the first highest peak of the distribution of T_s (Fig. S1A–C) and the period $t \leq T^*$ as its *early growth phase*. We find $T^* = 180$ days for handsets, $T^* = 4$ months for automobiles, and $T^* = 7$ days for apps. The differences in T^* across the three systems agree with our intuition of the typical lifecycle differences among handsets, automobiles, and mobile apps.

S1.3 Early Growth Pattern

In Fig. S1D–F, we show the early growth pattern for the selected products in Fig. 2. Instead of normalizing the impacts with $I(1)$ (the number of users on the released date), we show the

original impact dynamics for the products, finding they all follow power law growth patterns. Interestingly, we discover that handsets and automobiles with high $I(1)$ are not associated with high power law exponent η . To systematically study this phenomenon, we plot the relationship between $I(1)$ and η in Fig. S1G–I, finding the phenomenon is system-dependent. While $I(1)$ and η are weakly but negatively correlated for handsets and automobiles, suggesting products that registered the most sales tend to have a slower build up in its sales, the two parameters are largely independent of each other for smartphone apps. This indicates that $I(1)$ and η may not be driven by the same mechanism, pertaining to different processes governing substitution dynamics. Notice that the number of products shown in Fig. S1 represents a rather substantial fraction of visible products within the system given that impact typically follows fat-tail distributions (Fig. S3). Indeed, most products have relatively small impacts, limiting the number of samples to study their impact dynamics properly.

To systematically study how well the early growth patterns could be fitted as power laws, for each product with enough statistics in the three systems, we quantify the goodness of fit by measuring the coefficient of determination (R^2). In Fig. S4A, we show the complementary cumulative distribution of R^2 , finding most of the curves can be well fitted as power laws. We also find, while the vast majority of products follow power law growth patterns, for some of the cases in each of the three systems, their impacts start to saturate at a rather early stage, sometimes earlier than T^* (Fig. S4B–D), corresponding to cases with a low R^2 , as shown in Fig. S4A.

S2 Substitution in Handset Datasets

One important common characteristic among three studied systems discussed in S1 is that they evolve by substitutions. Although there has been a profusion of empirical studies with the recent big data explosion, particularly those emerging from online domains, tracing and measuring

substitution patterns empirically have remained as a difficult, often elusive task. This may seem puzzling given the fact that models that can be used to describe substitution processes have existed for over a century (29–31, 34–36). Here we explain this situation by highlighting the key challenges that have long prevented researchers from empirical studies of substitutions, and how mobile phone datasets used in our study offer a unique opportunity to allow us to present among the first empirical evidence on substitutions.

S2.1 Challenges in empirically studying substitutions

The lack of empirical knowledge about substitution patterns is rooted in the significant, systematic challenges in collecting adequate datasets to empirically trace and measure substitution patterns:

Challenge One (*C1*): Substitutions depend strongly on time, often signaling the beginning and end of a lifecycle. Hence measuring substitutions requires longitudinal datasets that can cover a longer time period than a typical lifecycle, rendering obsolete many datasets, particularly those emerging from online settings, which span comparably or less than the typical lifetime (37, 38).

Challenge Two (*C2*): Substitution implies a competitive process, in which we choose one or few out of many alternatives to substitute for. Therefore, understanding substitutions requires us to observe both the substituted ones and the alternatives. Yet studies that are potentially relevant typically involve a single (11, 31, 39) or an incomplete set (35, 40) of substitutes, hence inevitably focus on the substituted ones, by implicitly ignoring the alternatives. This is further confounded by the well-known heterogeneity in complex systems as popularity follows a fat-tailed distribution (41–45).

Challenge Three (*C3*): Substitutions involve both substitutes and the incumbents. To observe substitutions we need to go beyond aggregated records to obtain individual level substi-

tution histories. Otherwise, even in cases where datasets (occasionally) met $C1$ and/or $C2$, it is nearly impossible to infer accurately which substitutes for which (46, 47).

Here we take advantage of the increasing availability of rich databases in a ubiquitous setting, allowing us to systematically alleviate and combat all three aforementioned challenges: mobile telephony in the telecommunication sector. Indeed, mobile phones have existed with high penetration in developed countries for over a decade. Since the average usage time of a phone is less than two years, it offers an observation window that far exceeds the typical life cycle of the substitutes, in doing so eliminating $C1$. Carriers for billing purposes monitor all handsets that have ever operated within the network, ensuring the completeness in the set of substitutes we study ($C2$). Anonymized phone numbers together with their portability across devices provide individual traces for adoption and discontinuance histories, offering an excellent proxy of substitutions at an individual level within a societal-scale population, hence resolving $C3$.

S2.2 Substitution Patterns in Handset dataset

We start by analyzing the macroscopic properties of the mobile phone dataset and measure the total number of active handsets/users in the system as a function of time (Fig. S5). We find both quantities saturate to a constant $N = 2.5 \times 10^6$, indicating that the system reaches to a dynamical equilibrium around 2011. It also suggests that each individual in the dataset is holding one single product on average at a time, enabling us to compile the substitution timeline for each user accordingly (see Fig. S6A for an illustration) and generate a dynamic network characterizing substitution patterns among handsets.

In order to uncover the basic properties of this substitutive system, we specifically focus on an aggregated network capturing substitution patterns among 558 handsets within a six-month period 01/01/2014 — 06/01/2014, of which we have shown the backbone in Fig. 3A. While

the network has a large average degree ($\langle K \rangle = 73.6$), suggesting handsets are substituted by a considerable number of other handsets, the in (out)-degree distribution of the network follows a fat-tailed distribution (Fig. S6B), indicating a high heterogeneity in substitution selections. We also measured the distribution of substitution flows between two handsets, represented by the weight of the links, finding the distribution also follows a fat-tailed distribution (Fig. S6C). In addition to the structural complexity depicted in Fig. S6B–C, substitution patterns are characterized by a high degree of temporal variability. Indeed, the system turns into widely different configurations every year (Fig. S7B–E), driving the rise and fall patterns of handset popularities (Fig. S7A),

S3 Existing Models

Over the past century, a considerable number of studies have been devoted to understanding spreading and contagion processes from a wide range of fields: from economics and sociology (48–53) to computational social science (54–57), from epidemiology (6, 58) to computer science and physics (59–66), giving birth to an immense number of mathematical models. In this section, we classify the existing models into four different categories. For each of the category, we select among the most relevant models to show their analytical solutions and demonstrate why *none* of the them can explain the power law growth patterns observed in our data. We will also discuss the relationship between a few selected models and the Minimal Substitution model (*MS* model) proposed in our paper. In Table. S1 we summarize for several existing models their analytical solutions and early behaviors.

S3.1 Diffusion of Innovations Models

S3.1.1 Logistic Model

The logistic model (also known as the *SI* model in epidemiology) is widely utilized to model population growth, product adoption (67) and epidemic spreading (6), with application in many fields. In the context of production adoption, people from a conservative system are categorized as two different types: potential users and current users. In each time step, potential users are affected by current users to adopt the product with a certain probability q . With time, the attractiveness of the product decays, as the product have been adopted by all potential users in the system, the number of current users approaches a constant I^∞ , capturing the ultimate impact of the product. This process can be expressed in a rate equation:

$$\frac{dI_i}{dt} = q_i I_i (1 - I_i / I_i^\infty), \quad (1)$$

yielding

$$I_i(t) = \frac{I_i^\infty}{1 + e^{-q_i(t-\tau_i)}}, \quad (2)$$

where I_i^∞ , q_i and τ_i capture the ultimate impact, longevity, and immediacy of a product, respectively. By taking $t \rightarrow 0$, we obtain the early growth pattern predicted by the model, corresponding to an exponential growth pattern:

$$I_i(t)|_{t \rightarrow 0} = I_i(0)e^{q_i t}, \quad (3)$$

where

$$I_i(0) = \frac{I_i^\infty}{1 + e^{q_i \tau_i}}, \quad (4)$$

captures the number of initial users of the product.

S3.1.2 Bass Model

First proposed by Frank Bass in 1969, the Bass model (9, 68) is widely used in marketing, management science and technology forecasting. It describes the process through which new prod-

uct are adopted by mass populations. The Bass model classifies the adopter into two groups: innovators who are mainly influenced by the mass media and imitators who adopted the product through the *word of mouth* effect. Mathematically, this can be expressed as

$$\frac{dI_i}{dt} = (p_i + q_i I_i / I_i^\infty)(I_i^\infty - I_i), \quad (5)$$

where the impact of a product I is defined as the number of users. p describes the probability for innovators to adopt the product, reflecting a social influence effect that is independent of the current product impact. q captures the imitation process, where potential users are influenced by previous users with probability q . I^∞ defines the ultimate impact of the product, capturing total number of users of the product. Solving the model yields

$$I_i(t) = I_i^\infty \frac{1 - e^{-(p_i+q_i)t}}{1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}}. \quad (6)$$

By taking $t \rightarrow 0$, we obtain the early growth pattern of the model,

$$I_i(t)|_{t \rightarrow 0} = I_i^\infty p_i t \quad (7)$$

which corresponds to a linear growth pattern, different from the non-integer power law growth observed in our data.

S3.1.3 Gompertz Model

The Gompertz model, named after Benjamin Gompertz, was first proposed to model mortality (69). It has also been widely adopted to model market impact and product penetration (70, 71).

The model can be formulated as

$$\frac{dI_i}{dt} = q_i I_i \ln(I_i^\infty / I_i). \quad (8)$$

Solving the equation, we have:

$$I_i(t) = I_i^\infty e^{-e^{-(a_i+q_i t)}}. \quad (9)$$

The equation predicts that the product initiates from a finite number of users $I_i(0) = I_i^\infty e^{-e^{-a_i}}$, and grows exponentially at early stage:

$$I_i(t)|_{t \rightarrow 0} = I_i(0)e^{e^{-a_i}q_i t}. \quad (10)$$

S3.2 Substitution Models

S3.2.1 Fisher-Pry Model

The Fisher-Pry Model is considered as one of the earliest substitution models (31). It has been applied to model different substitution processes, from Synthetic/Natural Rubbers to Plastic/Natural Leathers. The model focuses on a two-product system, describing how a new product substitutes for an old one. Since only two products are considered, the model can be considered as mathematically similar to the logistic model, predicting a logistic growth pattern of the new product. Therefore, the early growth pattern predicted by Fisher-Pry model is exponential as well.

S3.2.2 Lotka-Volterra Competition Model

The Lotka-Volterra Competition (LVC) model, is frequently used to model population dynamics in biological systems. Along with its many variants, the model is widely applied to describe interaction dynamics: from species interactions to parasitic and symbiotic relations to technology competitions (30, 34, 72–74).

Here, we study the original version of the LVC model for a two-competitor system. Note that the model can be easily generated to a multi-product system, but the original LVC model is sufficient to illustrate the early behavior of products. The model contains two non-linear differential equations, capturing the population dynamics of the system:

$$\begin{aligned} \frac{dN_i}{dt} &= \frac{q_1 N_i}{K_i} (K_i - N_i - \alpha_2 N_j) \\ \frac{dN_j}{dt} &= \frac{q_2 N_j}{K_j} (K_j - N_j - \alpha_1 N_i). \end{aligned} \quad (11)$$

In (11), we denote N_i and N_j as the number of current users of the incumbents and substitutes. The competition between products are captured by coupling terms in both equations and controlled by the positive coefficients α_1 and α_2 . Note that α_1 and α_2 do not necessarily equal to each other, indicating that the influences of the two products on each other can be different. K_i and K_j capture the market size of each technology, equivalent to their ultimate impacts in the absence of competition.

To obtain the early growth pattern of an entrant, we assume that the incumbent dominates the market when the new entrant is introduced. We studied the asymptotic temporal behavior of N_j around the fixed point ($N_i = K_i, N_j = 0$), obtaining:

$$\frac{dN_j}{dt} = q_2 N_j - \frac{q_2 N_j}{K_j} \alpha_1 K_i, \quad (12)$$

where the $q_2 N_j$ term corresponds to an exponential growth of the substitute, and the $\frac{q_2 N_j}{K_j} \alpha_1 K_i$ term reflects the discontinuance of j due to the competition with i . By solving (12), we obtain the early growth pattern of the substitutes:

$$N_j(t)|_{t \rightarrow 0} = N_j(0) e^{q_2(1 - \frac{\alpha_1 K_i}{K_j})t}, \quad (13)$$

which is an exponential function. We can also attain another exponential growth of j 's impact by solving $\frac{dI_j}{dt} = q_2 N_j$:

$$I_j(t)|_{t \rightarrow 0} = I_j(0) e^{q_2(1 - \frac{\alpha_1 K_i}{K_j})t}. \quad (14)$$

S3.2.3 Norton-Bass Model

The Norton-Bass (NB) model was proposed by Norton and Bass in 1987 aiming at describing multi-generation diffusion processes (35, 68). Inspired by the seminal Bass model (9), the NB model consider the penetration of technology that evolves rapidly in successive generations.

The NB model consists of k nonlinear equations describing the sales of k -generation technologies with continuous repeat purchasing. For simplicity, here we consider a system of two

generations, a more complex k -generation case can be generalized in a straightforward manner from the following results. According to the NB model, we have:

$$N_i = K_i F_i(t_i) - K_i F_i(t_i) F_j(t_j) \quad (15)$$

$$N_j = K_j F_j(t_j) + K_i F_i(t_i) F_j(t_j),$$

where t_i and t_j represent the age of the old generation production (i) and the new product (j). K_i and K_j capture the market capacity of the products. N_i and N_j measure the product sales. Notice that the original NB model is designed for the product with continuous, repeated purchases, the sales at a given time can be approximated as the current number of users of a given product. The function $F_g(t_g)$ takes the following form:

$$F_g = \frac{1 - e^{-(p_g+q_g)t_g}}{1 + \frac{q_g}{p_g} e^{-(p_g+q_g)t_g}}, \quad (16)$$

which is derived from the Bass model (see Eq. 5 and Eq. 6). The interaction between products is captured by the coupling terms $K_i F_i(t_i) F_j(t_j)$, without which the behavior of the products follows the original Bass model. To understand the behavior of the NB model, we take the limit $t \rightarrow \infty$, obtaining $N_i = 0$ and $N_j = K_i + K_j$, which indicates that the new product will take over the entire market.

From (15), we derive the asymptotic temporal behavior of j around $t_j \rightarrow 0$, yielding:

$$N_j(t_j)|_{t_j \rightarrow 0} = p_j \left(K_j + K_i \frac{1 - e^{-(p_i+q_i)t_\Delta}}{1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t_\Delta}} \right) t_j, \quad (17)$$

where $t_\Delta = t_i - t_j$ measures the age difference of the products. Eq. 17 indicates that the early impact dynamics of j can be approximated by linear growth patterns, hence different from the non-integer power law growth observed in our data.

S3.3 Epidemic Models

S3.3.1 SIR Model

Epidemic models are another class of models that can be generalized to describe substitutions. One of the most famous models in this class is the SIR model (3, 6, 58). Here people are

classified into three groups: S represents the susceptibles, measuring the number of people who are susceptible to adopt a product; I , the infectious, measures the number of people who currently use the product; and R , the recovered group captures people who had bought the product previously, but have discontinued using it. In each time step, a current user “infects” a susceptible user with probability β , and at the same time, the user may abandon the product (recover) with certain probability γ . To avoid confusion over I as impact throughout the paper, here we use A to represent the number of current users, corresponding to the quantity that is typically described as I in the SIR model. Mathematically, the model could be expressed as a set of ordinary differential equations,

$$\begin{aligned}\frac{dS}{dt} &= -\frac{\beta AS}{N_0} \\ \frac{dA}{dt} &= \frac{\beta AS}{N_0} - \gamma A \\ \frac{dR}{dt} &= \gamma A\end{aligned}\tag{18}$$

where N_0 captures the total number of people in the system, and the impact of the product can be obtained through its definition: $I(t) \equiv A(t) + R(t)$. The model does not have a closed form solution, but we can approximate the earlier behavior of the growth pattern analytically, finding that it follows an exponential growth at $t \rightarrow 0$:

$$A(t)|_{t \rightarrow 0} = A(0)e^{(\beta-\gamma)t}.\tag{19}$$

We can also derive the early growth pattern of I , which also follows exponential growth:

$$I(t)|_{t \rightarrow 0} = I(0)e^{(\beta-\gamma)t}\tag{20}$$

In fact, although the entire dynamic of other epidemic models (SIS, SIRS) are different from the SIR model, their early growth patterns follow the same exponential growth pattern. For a more comprehensive review of this body of literatures, refer to Ref. (3).

S3.3.2 Multiple Epidemics Model

Multiple Epidemics model was previously proposed to describe competitions among diseases. This type of models has been generalized to understand scientific paradigms shifting (75, 76). The original model focuses on a two-dimensional lattice where each site represents a particular user. In each time step, one attempts the following two moves: 1) A random site i is selected and i will randomly choose one of its four neighbors j . If i has not used j 's current product before, she/he would adopt the product; Otherwise, the system remains the same. 2) With probability α , another random site k is selected and a newly introduced product will be assigned to the node occupying site k .

Monte Carlo simulation shows that in this model, the early growth pattern of product impact is determined by the lattice dimension. The model predicts that a product's impact growth rate increases linearly at its early stage in a two-dimensional lattice, indicating impact follows a quadratic growth pattern at beginning. If we change the lattice assumption to a random graph the model predicts that the growth pattern follows an exponential growth. Therefore the class of models lacks the mechanisms to explain the divergent behavior in small t region predicted by the non-integer power law exponents.

S3.3.3 Sub-exponential Growth Model

While most epidemic models predict exponential growth, recent sub-national epidemiological data at the level of counties or districts offered new observations that infectious diseases spreading via close contacts (sexually-transmitted infectious diseases, smallpox, and Ebola) exhibit sub-exponential early growth patterns (32, 33, 77). As pointed out (33, 78), the observed sub-exponential early growth patterns are consistent with the formalism:

$$\frac{I_i(t)}{dt} = r_i I_i(t)^{p_i}, \quad (21)$$

where r captures the growth rate of the disease and p is the “deceleration of growth” parameter. When $p = 0$, a linear growth pattern is expected, whereas $p = 1$ would generate an exponential growth pattern. Models with similar forms have also been applied to describe innovation diffusions (see review (46)). While (21) does not offer any mechanistic explanation for the observed power law growth, these examples show that the power law growth patterns we observed in this paper may possibly extend beyond substitutive systems.

S3.4 Network Growth Models

Network growth models represent a well-known branch of models that are often associated with power laws (27, 79–81). Next, we will first discuss two types of network growth models: 1) Evolving network models that explain degree dynamics and heterogeneity, such as the BA model (45) and the fitness model (79, 80); 2) Network densification models that explain the growth in the number of nodes and links (27, 81). We will then demonstrate that power laws generated by network models vis-a-vis what is observed in substitutive systems pertain to fundamentally different processes.

S3.4.1 Evolving Network Models

The fitness model (also known as Bianconi-Barabási model) was proposed to model the evolution of a competitive networked system (79, 80). At each time step, new products (represented by nodes) are introduced at a constant rate. They link with existing nodes with probability

$$\Pi_i \propto \eta_i I_i(t), \quad (22)$$

where fitness parameter η_i quantifies the likelihood of product i to be adopted by users, $I_i(t)$ corresponds to the product impact, i.e., the degree of node i , capturing the well-known preferential attachment mechanism. If we set $\eta_i = 1$ for all nodes i , the model reduces to the BA model (45).

The fitness model predicts that, the node dynamics follow a power law growth, with the exponent governed by fitness:

$$I_i \propto t^{\frac{\eta_i}{C}}. \quad (23)$$

C is a global parameter in the interval $(\eta_{max}, 2\eta_{max}]$, which can be obtained from the following equation:

$$1 = \int_0^{\eta_{max}} d\eta P(\eta) \frac{1}{\frac{C}{\eta} - 1}, \quad (24)$$

where $P(\eta)$ is the distribution of η . Therefore, the fitness model can predict a power law growth, but only for exponents that are in the interval $[0.5, 1)$. Notice that $\eta = 0.5$ corresponds to the prediction of the BA model, which can be treated as a special case of the fitness model in this regard (45).

S3.4.2 Network Densification Models

The seminal work by Leskovec, Kleinberg and Faloutsos (81) outlines another mechanism for power law growth to emerge in the network context: densification in networks follows power law growth patterns due to the fact that the number of nodes and edges grows as power laws. This class of models also includes a recent variant called the NetTide model (27), which describes power law growth patterns in the number of users in social networking sites, such as WeChat and Weibo. The NetTide model focus on a single product, where existing users invite non-users with certain time-varying probability: $\frac{dI_i}{dt} = \frac{\beta_i}{t^\theta} I_i(t)(I_i^\infty - I_i(t))$. By setting $\theta = 1$, the model predicts that the early growth pattern of a product follows a power law: $I \propto t^\beta$. Similar equations have also been proposed to understand technology penetration (82, 83). This class of network densification models usually focuses on the growth patterns of one single product. While it is not clear how, and if at all, one may generalize the model to describe systems containing multiple products, the ability of these models to predict power law growth raises an interesting question: how do they relate to the observed power law patterns documented in our

paper? Next we show, the growth patterns predicted in network models described in this section pertain to fundamentally different processes than what we observed, hence can not be adapted to explain our phenomena.

S3.4.3 Relationship between the *MS* Model and network growth models

The key for the two classes of network models described in Sec. S3.4.1 and S3.4.2 to generate power law growth is because of the growth of the system. That is, the number of nodes and edges increases with time as a power law. This raises an interesting question: Can the power law growth pattern we observed in substitutive systems be explained by the expansion of the system? Indeed, as we show in Fig. S5, while our system converges quickly to a relatively stable system, it still grows slightly over time with addition of new users. To answer this question, we therefore studied a stable system by removing the contributions from new subscribers in our dataset.

The new system is comprised of 1.64 Million people and their usage patterns in a two-year time window from 2010 to 2012. Each user uses only one product at a time in this period (Fig. S5). For each of the product released in the two year period, we define its impact $I(t)$ as the total number of users among the population. Because there is no growth in the number of users in our system, all models described in Sec. S3.4.1 and S3.4.2 would break down. Hence it begs the question of whether the power law growth in substitution patterns persists in the absence of system growth. Interestingly, after eliminating the effect of growth in the number of users, we find the impact dynamics of individual handsets remain intact, following again a clear power law growth pattern (Fig. S8). This finding indicates that the power law growth pattern in the substitutive system are not due to system expansion. Rather it pertains to mechanisms that are specific to substitutive systems.

In the next section, we present in detail our Minimal Substitution (*MS*) model. The model

not only allows us to offer a mechanistic explanation for the observed power law growth pattern in substitutive systems, but also accurately captures the entire lifecycle of product impacts, collapsing constituents from a wide range of domains into a single universal curve, documenting a remarkable degree of regularity underlying the ubiquitous substitutive systems.

S4 Minimal Substitution (*MS*) model

S4.1 Model Description

In the proposed model, we consider a conservative system comprised of N_0 users, where each individual uses one handset at a time. With time, new handsets are introduced into the system at a constant rate ρ , prompting users to substitute their incumbent products with new innovations. In each time step, an individual substitutes another handset j for her/his current handset i with probability $\Pi_{i \rightarrow j}$:

$$\Pi_{i \rightarrow j} = \lambda_{ij} N_j \frac{1}{t_j}, \quad (25)$$

where N_j captures the popularity of the handset j and t_j measures its age. The factor N_j in (25) captures the *preferential attachment* mechanism (43–45), suggesting that people tend to adopt handsets of higher popularity. The $1/t_j$ factor corresponds to the *recency* mechanism, uncovered by the data collapse documented in Fig. 3F. Indeed, while two-years is the typical age of a handset when it is substituted by other products, the distribution of the age of substitutes peaks much earlier (Fig. S9), indicating that user prefers handsets that are released more recently. The factor $\lambda_{i \rightarrow j}$ reflects the inherent *propensity* between two given products i and j , capturing the heterogeneous nature in the likelihood of substitutions. Note that all factors in (25) are empirically validated and motivated in the main text, hence (25) represents a minimal model that brings together all mechanisms we know to date governing substitutions.

S4.2 Solving the MS model using the Master Equation

Given (25), the popularity dynamics of an individual handset can be expressed in the master equation formalism:

$$\begin{aligned}\frac{dN_i}{dt_i} &= \sum_k \Pi_{k \rightarrow i} N_k - \sum_j \Pi_{i \rightarrow j} N_i \\ &= \sum_k \lambda_{k \rightarrow i} N_k N_i t_i^{-1} - \sum_j \lambda_{i \rightarrow j} N_i N_j t_j^{-1}.\end{aligned}\quad (26)$$

Defining fitness as $\eta_i \equiv \sum_k \lambda_{k \rightarrow i} N_k$ and longevity τ_i as $\tau_i \equiv 1 / \sum_j \Pi_{i \rightarrow j}$, we have

$$\frac{dN_i}{dt_i} = \eta_i N_i t_i^{-1} - N_i / \tau_i. \quad (27)$$

Next, we show that both η_i and τ_i are time-independent parameters in a stationary system. Because the propensity parameter $\lambda_{k \rightarrow i}$ between two products is independent of the popularity of a product, i.e. $\lambda_{k \rightarrow i}$ is independent of N_k and N_i , allowing us to write:

$$\begin{aligned}\eta_i &\equiv \sum_k \lambda_{k \rightarrow i} N_k \\ &\approx \sum_k \lambda_{k \rightarrow i} p(\lambda_{k \rightarrow i} | i) \sum_k N_k \\ &= N_0 \sum_k \lambda_{k \rightarrow i} p(\lambda_{k \rightarrow i} | i) = N_0 \lambda_i^{\leftarrow}.\end{aligned}\quad (28)$$

We discover that η_i only depends on two time-independent parameters: N_0 , the total number of users in the system, and λ_i^{\leftarrow} , the average propensity from all other handsets *towards* i , indicating that η_i is also time-independent.

We repeat the calculations above for the longevity τ , obtaining:

$$\begin{aligned}1/\tau_i &\equiv \sum_j \Pi_{i \rightarrow j} \\ &= \sum_j \lambda_{i \rightarrow j} N_j t_j^{-1} \\ &\approx \sum_j N_j t_j^{-1} \sum_j \lambda_{i \rightarrow j} p(\lambda_{i \rightarrow j} | i) \\ &= \lambda_i^{\rightarrow} \sum_j N_j t_j^{-1}.\end{aligned}\quad (29)$$

In order to study the property of $\sum_j N_j t_j^{-1}$, we consider a quantity $N^{(\mathcal{T})}$ measuring the total number of users who are holding handsets of age \mathcal{T} . Because $N^{(\mathcal{T})}$ is time-independent for any given handset age \mathcal{T} in a stationary system, we can rewrite $\sum_j N_j t_j^{-1}$ as $\sum_{\mathcal{T}=1}^{\infty} N^{(\mathcal{T})} \mathcal{T}^{-1}$. By defining $M_0 \equiv \sum_{\mathcal{T}=1}^{\infty} N^{(\mathcal{T})} \mathcal{T}^{-1}$, we obtain:

$$\begin{aligned}
1/\tau_i &\equiv \sum_j \Pi_{i \rightarrow j} \\
&\approx \sum_j N_j t_j^{-1} \sum_j \lambda_{i \rightarrow j} p(\lambda_{i \rightarrow j} | i) \\
&= M_0 \lambda_i^{\rightarrow}.
\end{aligned} \tag{30}$$

Eq. (30) reveals that the longevity τ_i is inversely proportional to two time-independent parameters: M_0 , a global parameter capturing the *effective popularity* of handsets in the system and λ_i^{\rightarrow} , the average propensity *from* i to all other handsets, thus demonstrating the time-independency of τ_i . Note that we have made approximations in (28) and (30), by assuming that λ and N are independent. In Sec. 4.3, we will demonstrate the time-independence of the parameters without making these approximations by studying a continuous formalism of the *MS* model.

S4.3 Mapping the system into a Continuous Space

In this section we discuss a continuous formalism of the *MS* model by mapping handsets into a property space, enabling us to rigorously show the time-independent nature of the model parameters. To do this, we introduce a continuous vector φ to represent a given handset's functions and properties. For any handset in the system, we assume its growth dynamic is determined by φ . Hence the product's popularity could be denoted by $N(\varphi, t)$, with t capturing the handset's current age, and φ corresponding to its properties. In this continuous framework, an individual substitutes a handset (φ', t') for another product (φ, t) with probability:

$$\Pi(\varphi, t \rightarrow \varphi', t') = \lambda(\varphi, \varphi') N(\varphi', t') t'^{-1}, \tag{31}$$

where λ is a function of φ and φ' , capturing the propensity between the products. Since the total number of people in the system is a constant (N_0), the popularity of the handsets in the system satisfies the following condition:

$$N_0 = \rho \int_{\varphi} P(\varphi) d\varphi \int_0^{\infty} N(\varphi, t) dt, \quad (32)$$

where ρ measures the release rate of new handsets and $P(\varphi)$ corresponds to a distribution, from which a new handset's φ is drawn.

The popularity dynamic of any individual handset follows the master equation:

$$\frac{\partial N(\varphi, t)}{\partial t} = \rho \int_{\varphi'} P(\varphi') d\varphi' \int_0^{\infty} [\Pi(\varphi', t' \rightarrow \varphi, t) N(\varphi', t') - \Pi(\varphi, t \rightarrow \varphi', t') N(\varphi, t)] dt'. \quad (33)$$

Inserting (31) into (33), we have

$$\begin{aligned} \frac{\partial N(\varphi, t)}{\partial t} &= \rho N(\varphi, t) t^{-1} \int_{\varphi'} P(\varphi') d\varphi' \lambda(\varphi', \varphi) \int_0^{\infty} N(\varphi', t') dt' \\ &\quad - \rho N(\varphi, t) \int_{\varphi'} P(\varphi') d\varphi' \lambda(\varphi, \varphi') \int_0^{\infty} t'^{-1} N(\varphi', t') dt' \\ &= \eta(\varphi) N(\varphi, t) t^{-1} - N(\varphi, t) / \tau(\varphi), \end{aligned} \quad (34)$$

where the handset's fitness is defined as:

$$\eta(\varphi) \equiv \rho \int_{\varphi'} P(\varphi') d\varphi' \lambda(\varphi', \varphi) \int_0^{\infty} N(\varphi', t') dt', \quad (35)$$

and its longevity as :

$$\tau(\varphi) \equiv \frac{1}{\rho \int_{\varphi'} P(\varphi') d\varphi' \lambda(\varphi, \varphi') \int_0^{\infty} t'^{-1} N(\varphi', t') dt'}. \quad (36)$$

We find both parameters are time-independent.

S4.4 Solving the Master Equation

By change of variable $f_i = \ln N_i$, we rewrite (26) as:

$$\frac{df_i}{dt_i} = \eta_i t_i^{-1} - 1/\tau_i. \quad (37)$$

By solving the equation, we arrive at:

$$f_i = \eta_i \ln(t_i) - t_i/\tau_i + C_i, \quad (38)$$

and

$$N_i(t) = h_i t^{\eta_i} e^{-t/\tau_i}. \quad (39)$$

$h_i \equiv e^{C_i}$ corresponds to the anticipation factor. Since the impact of a handset (I) measures its total number of adopters, the impact dynamics can be obtained by solving the following equation:

$$\frac{dI_i(t)}{dt} = \eta_i N_i(t) t^{-1}. \quad (40)$$

By inserting (39) into (40), we obtain:

$$I_i(t) = h_i \eta_i \tau_i^{\eta_i} \gamma_{\eta_i}(t/\tau_i), \quad (41)$$

where γ corresponds to the incomplete gamma function $\gamma_z(t) \equiv \int_0^t x^{z-1} e^{-x} dx$. Interestingly, (41) suggests that the impact of a handset should saturate to a constant. Indeed, If we take the limit $t \rightarrow \infty$, the formula predicts the ultimate impact of a handset:

$$I_i^\infty = h_i \Gamma(\eta_i + 1) \tau_i^{\eta_i}, \quad (42)$$

where $\Gamma(z) \equiv \int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function.

S4.5 Maximum Likelihood Estimation of Model Parameters

In order to test the model performance, we need to estimate the best parameter set (h, η, τ) of each handset, simulate its impact dynamics through (41), and compare it with the empirical observation. To achieve this, let us imagine a non-homogeneous stochastic process $\{x(t)\}$, with $x(t)$ representing the number of new adoptions by time t , satisfying:

$$Prob(x(t+h) - x(t) = 1) = \lambda_0(x, t)h + O(h^2), \quad (43)$$

where $\lambda_0(x, t)$ is a time dependent rate parameter. Given an empirically observed set of N events $\{t_i\}$ within the time period $[0, T]$, where t_i indicates the moment when the product gets adopted the i^{th} time, the likelihood that the product's impact dynamics follows can be evaluated by the log-likelihood function:

$$\begin{aligned}\ln L &= \sum_{i=1}^N \ln(\lambda_0(i-1, t_i)) - \int_0^T \lambda_0(x(t), t) dt \\ &= \sum_{i=1}^N \ln(\lambda_0(i-1, t_i)) - \sum_{i=0}^N \int_{t_i}^{t_{i+1}} \lambda_0(i, t) dt.\end{aligned}\quad (44)$$

To find $\lambda_0(x, t)$ in our system, we insert (39) into (40), yielding

$$\frac{dI_i}{dt} = h_i \eta_i t^{\eta_i - 1} e^{-t/\tau_i}. \quad (45)$$

Thus, in our system, we have $\lambda_0 = h\eta t^{\eta-1} e^{-t/\tau}$. By change of variable $H \equiv h\eta$ and $\nu \equiv 1/\tau$, we obtain the log-likelihood function:

$$\ln L = N \ln H + \sum_{i=1}^N (\eta - 1) \ln(t_i) + \sum_{i=1}^N (-\nu t_i) - H \nu^{-\eta} \gamma_{\eta}(\nu T). \quad (46)$$

The best-fitted parameters should maximize the log-likelihood function, satisfying the following equations,

$$\begin{aligned}\frac{\partial \ln L}{\partial H} &= 0 \\ \frac{\partial \ln L}{\partial \eta} &= 0 \\ \frac{\partial \ln L}{\partial \nu} &= 0.\end{aligned}\quad (47)$$

These equations lead to a set of non-linear equations,

$$\begin{aligned}H - N \nu^{\eta} \gamma_{\eta}^{-1}(\nu T) &= 0 \\ \sum_{i=1}^N \ln t_i + N \ln(\nu) - N \gamma_{\eta}^{-1}(\nu T) j_{\eta}(\nu T) &= 0 \\ - \sum_{i=1}^N t_i + N \eta \nu^{-1} - N \gamma_{\eta}^{-1}(\nu T) T [(\nu T)^{\eta-1} e^{-\nu T}] &= 0,\end{aligned}\quad (48)$$

where $j_z(x) \equiv \partial\gamma_z(x)/\partial z$ is the partial derivative of the incomplete gamma function on z . By solving (48), we are able to obtain the best fitted set of parameters (h, η, τ) for each product.

S4.6 Model Performance

We randomly select six handsets as examples to illustrate the model validation process. We learn the best fitted parameters (h, η, τ) for each of the product, insert them back into (41) and simulate the impact dynamic. We find the model not only well captures the early power growth pattern of each handset (Fig. S10A), but also accounts for their entire impact dynamics (Fig. S10B). In Fig. S10C, we show the impact trajectories of 100 different handsets, finding excellent agreement between the model predictions and empirical observations. The performance of the model does not rely on the particulars of the system. In Fig. S11AB, we show the impact dynamics of 70 automobiles and 200 apps, where again, the model captures impact trajectories accurately in both systems.

To systematically study the performance of the model, we calculate the coefficient of determination (R^2) for each fitting in all three systems and show the complementary cumulative distributions in Fig. S11C. We find that although the model perfectly capture accurately the impact trajectories for a vast majority of the products, there are occasional cases where the model prediction deviates from the data. In Fig. S10D, we show an example of such a case, indicating that impact dynamics with sudden discontinuities are not captured by our model.

S4.7 Linking short-term and long-term impacts

The *MS* model offers an intriguing linkage between a product's short-term impact and its long-term impact. By taking the derivative of (39), we obtain the moment t_i^* when the product's popularity reaches its peak,

$$t_i^* = \eta_i \tau_i. \tag{49}$$

By inserting (49) and (42) into (41), we discover that the handset's impact at t^* (short-term impact) and its ultimate impact I^∞ (long-term impact) can be connected by a simple equation:

$$\frac{I_i^\infty}{I_i(t_i^*)} = \Phi(\eta_i), \quad (50)$$

where Φ is a function of η , defined as $\Phi(\eta) \equiv \frac{\Gamma(\eta)}{\gamma_\eta(\eta)}$.

In order to test the formula empirically, we calculate the impact of each handset by 11/03/2014 (the last date in our dataset), denoting them as I^l . We specifically focus on 469 handsets whose I^l are close enough to their estimated ultimate impacts, satisfying the criterion: $\frac{I^\infty - I^l}{I^\infty} \leq 5\kappa$, where we choose $\kappa = 0.02$. To correct for the difference between the ultimate impact and I^l , we rescale I^l with $1 - \kappa$, obtaining an empirically estimated ultimate impact $I^e = I^l / (1 - \kappa)$. As for $I(t^*)$, we learn the three parameters (h, η, τ) for each product, calculate its t^* through (49) and find its empirical impact at t^* .

In Fig. S12A, we show I^e as a function of $I(t^*)$, finding they follow clear linear relationship, consistent with the prediction of (50). Furthermore, in Fig. S12B, we normalized I^e by $I(t^*)$, showing the ratio as a function of η . We find the slight increase trend in $\Phi(\eta)$ as a function of η is again accurately predicted by (50).

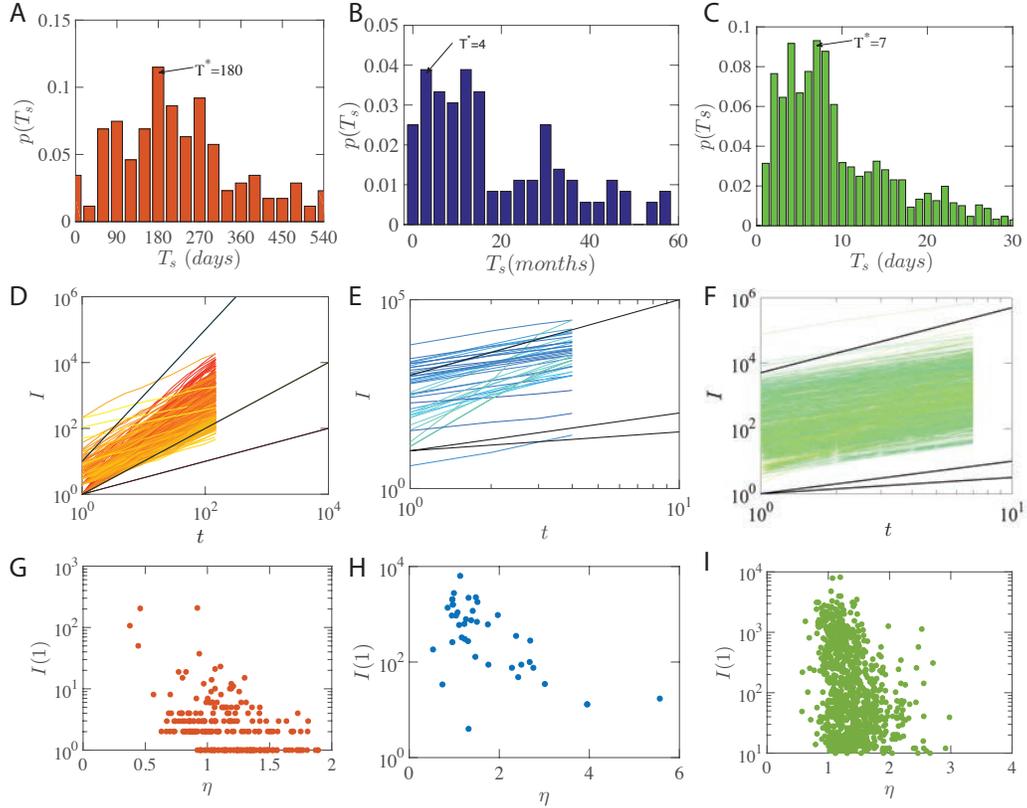


Figure S1: **Dataset description.** (A–C) Distribution of T_s for three datasets: handset (A), automobile (B) and smartphone app (C). We identify T^* as the position of the first highest peak of the distribution of T_s , finding $T^* = 180$ days for handsets, $T^* = 4$ months for automobiles and $T^* = 7$ days for apps. (D–F) Impact as a function of time for three datasets. The color of the line corresponds to the power law exponent of each handset. (G–I) The impact of the first time unit (first day for handset and app dataset, first month for automobile dataset) as a function of the power law exponent η characterizing the initial growth.

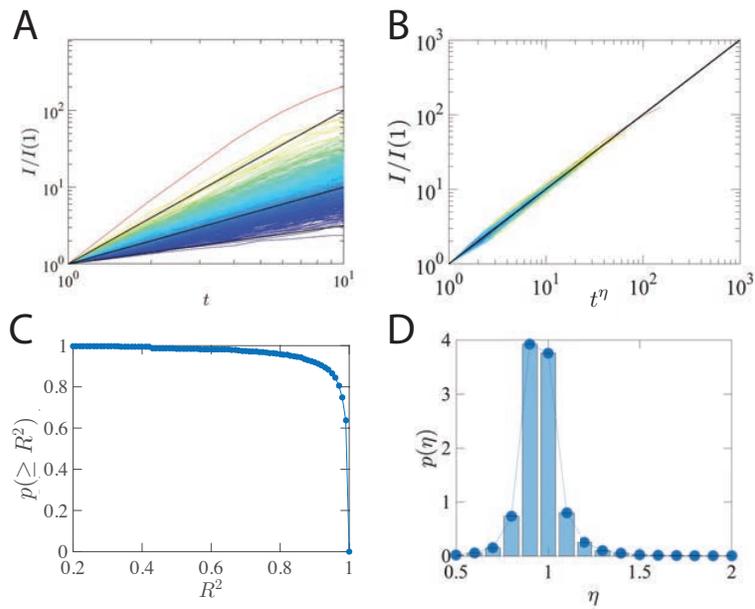


Figure S2: **Health apps dataset.** (A) We repeat the analysis in Fig. 2 on the Health App dataset, finding the impact dynamics follow the same power law growth patterns: $I(t) \sim t^\eta$. The color of the line corresponds to the power law exponent of each handset. The solid black lines are $y = x^{1/2}$, $y = x$, and $y = x^2$, respectively, as guides to the eye. (B) We rescale the impact dynamics plotted in (A) by t^η , finding all curves collapse into $y = x$. (C) The complementary cumulative distribution of R^2 , capturing how well the early growth patterns can be fitted as power laws. (D) Distribution of power law exponents $P(\eta)$ for curves in (A).

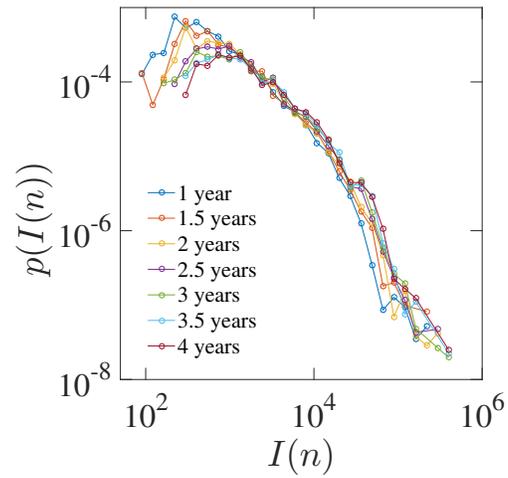


Figure S3: **Distribution of Impacts.** Distribution of the n -year impact ($I(n)$) of handsets. $I(n)$ is defined as the number of sales of a handset after released for n years.

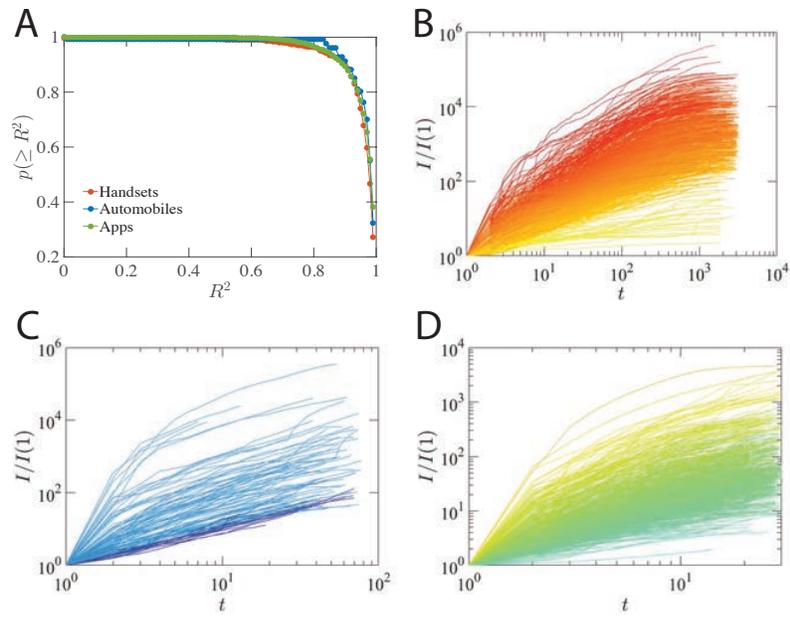


Figure S4: **Impact growth patterns.** (A) The complementary cumulative distribution of R^2 , capturing how well the early growth patterns can be fitted as power laws. (B–D) The impact growth patterns for the entire lifecycle across the three datasets.

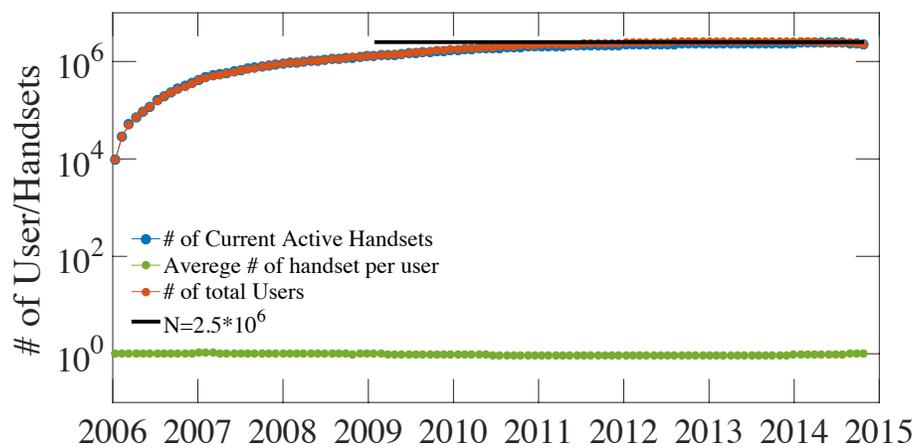


Figure S5: **Handset system as a substitutive system.** Number of current active handsets (current active users) as a function of time. We find both quantities saturate to a constant (black line). We also calculate the average number of handsets per user as a function of time, finding that people are holding one single handset at a time.

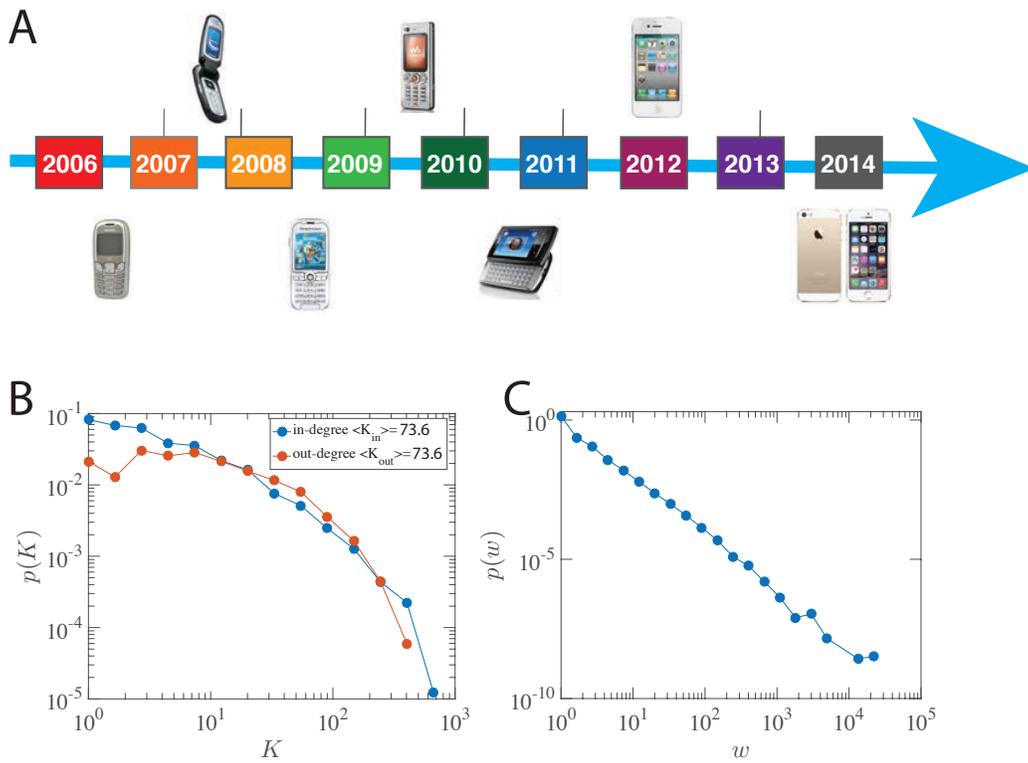


Figure S6: **Substitution network.** (A) Illustration of a usage timeline. (B) In-degree and out-degree distribution of the aggregated substitution network generated between January 2014 and June 2014. (C) Link weight distribution of the substitution network.

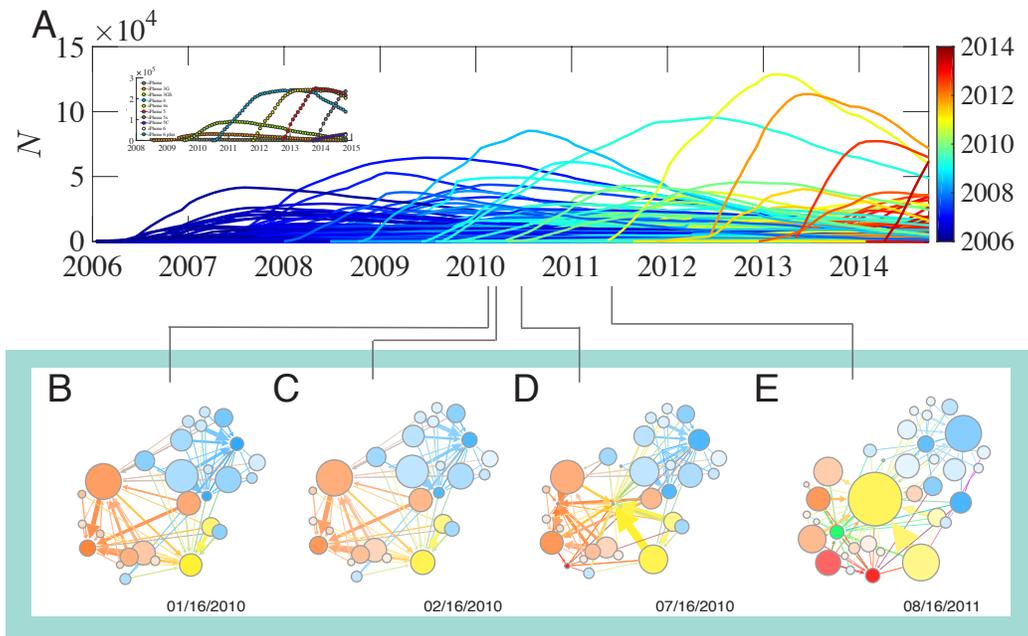


Figure S7: **Characterizing substitution patterns.** (A) Popularity of individual handsets (N) over time for products from Apple Inc (inset) and products from other companies (main). Each line represents a model of handset. The color of the lines correspond to the release dates of the products, shifting from blue to red. (B–E) Substitution network of top handsets in four selected snapshots. We selected for handsets who were ranked within top 10 based on their popularity. The size of the nodes captures the popularity of the handset. Handsets by manufacturers are shown in different node colors, which fade with the age of handsets. The weight of the link captures the number of substitution in a period of one month. We find, in addition to the complexity and heterogeneity depicted in Fig. 3A, substitution patterns are characterized by a remarkable amount of temporal variability.

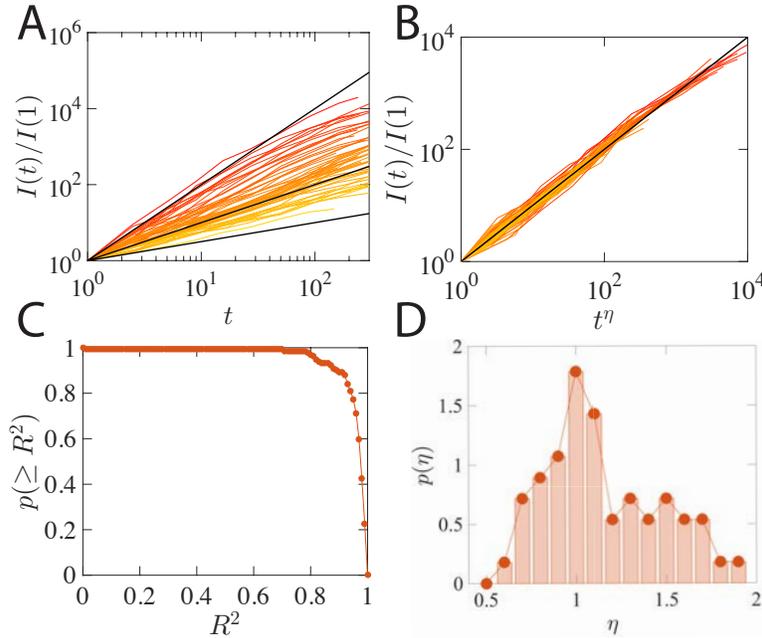


Figure S8: **Power law growth persists when the number of users stays constant.** To eliminate the influence of the network growth on the power law growth pattern, we explore a conservative system comprised of 1.64 Million users in a two-year time window (2010-2012). By removing potential contributions from new users, existing network models (Sec. 3.4) would predict the power law growth disappears. Yet, we find in our system the same power law growth patterns. **(A)** By repeating the analysis shown in Fig. 2, we study the growth pattern of 131 handsets released between 01/01/2010 and 06/01/2011 and selected 56 impact trajectories as power law examples. **(B)** We rescale the impact dynamics plotted in (A) by t^η , finding all curves collapse into $y = x$. **(C)** The complementary cumulative distribution of R^2 , capturing how well the early growth patterns can be fitted as power laws. **(D)** Distribution of power law exponents $P(\eta)$ for curves in (A).

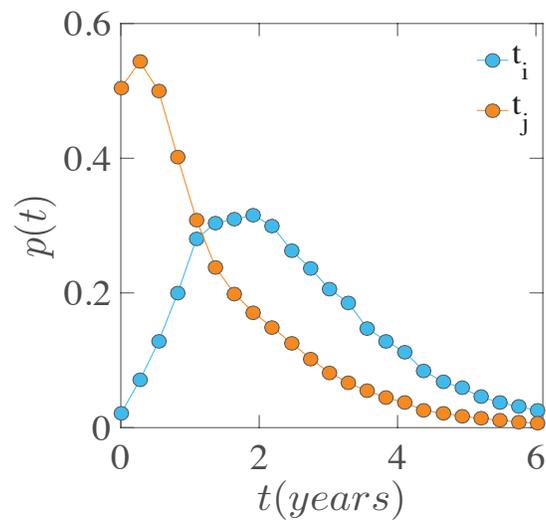


Figure S9: **Distribution of handset age.** For each substitution event in the system, we measure the age of the substitutes (t_j) and the incumbent (t_i). While the age distribution for the incumbent peaks around 2 years, the age of the substitutes peaks earlier, corroborating the *recency* mechanism uncovered in Fig. 3F.

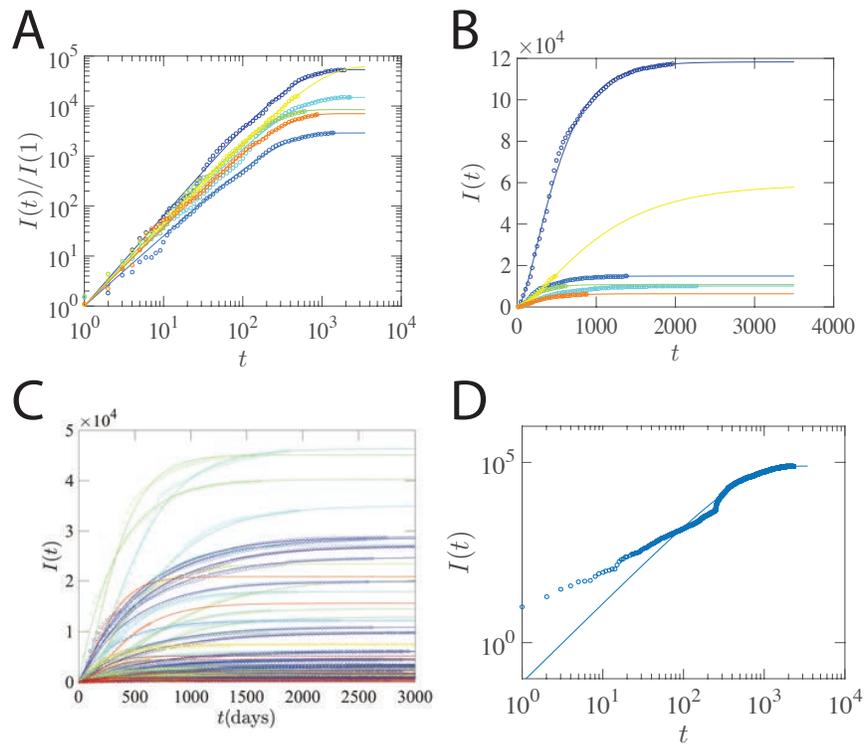


Figure S10: **Model validation.** (A–B) Comparison between the empirical observation (open circle) and model simulation (solid line) for 6 randomly selected handsets. We show the normalized impact dynamics in (A) and the impact dynamics in (B). (C) Impact trajectories of 100 randomly selected handsets. (D) An example where the model fails to capture the growth patterns due to sudden shifts and jumps in impact dynamics.

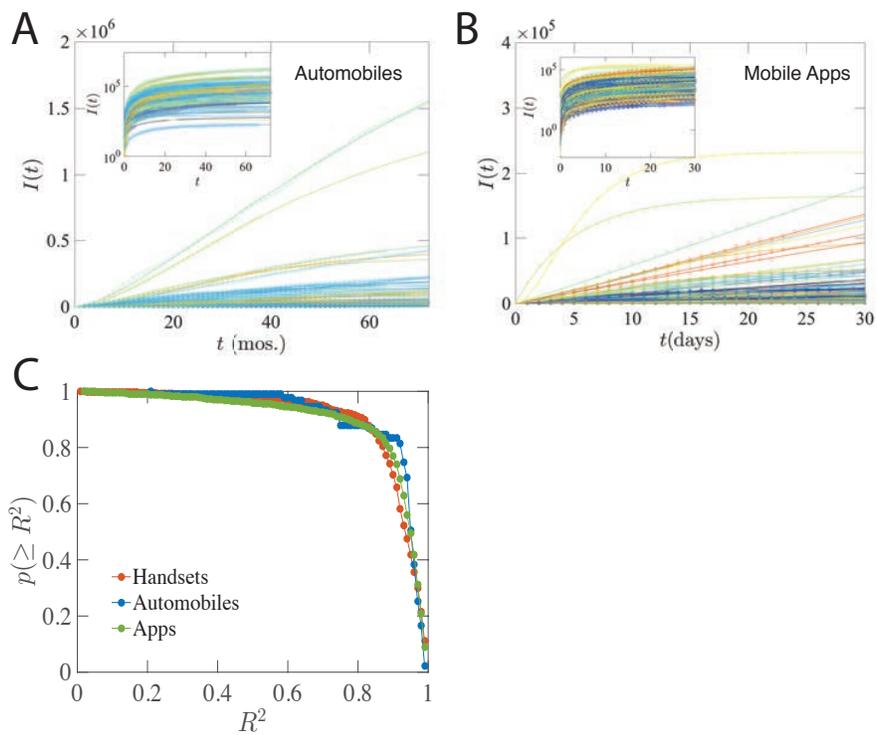


Figure S11: **Model validation for automobiles and apps.** (A–B) Impact trajectories of 70 automobiles (A) and 200 apps (B). (C) We fit each impact dynamics across three systems with the *MS* model and show the complimentary cumulative distribution of the R^2 of the fittings.

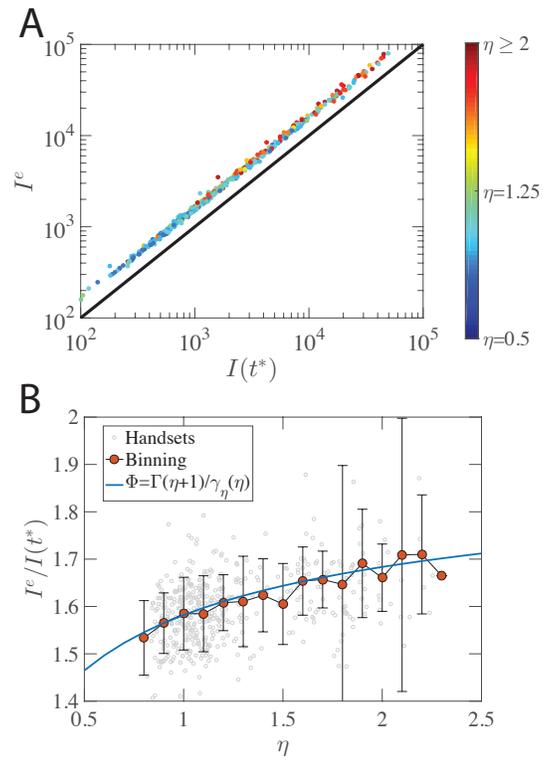


Figure S12: **Relationship between short-term and long-term impact.** (A) I^e as a function of $I(t^*)$. (B) The ratio between I^e and $I(t^*)$ as a function of the fitness η . Solid line corresponds to the function $\Phi(\eta)$.

| Model | Model Equation ($dI/dt =$) | Model Solution ($I =$) | Early Behavior ($I \sim$) |
|------------------------|--|---|-----------------------------|
| Logistic (31) | $qI(1 - I/I^\infty)$ | $I^\infty(1 + e^{-q(t-\tau)})^{-1}$ | $I(0)e^{qt}$ |
| Bass (9, 68) | $(p + qI/I^\infty)(I^\infty - I)$ | $I^\infty \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}$ | $I^\infty pt$ |
| Gompertz (69) | $qI \ln(I^\infty/I)$ | $I^\infty e^{-e^{-(a+qt)}}$ | $I(0)e^{e^{-a}qt}$ |
| SIR (3, 6, 58) | $\beta(I - R)(1 - I/I^\infty)$ | $N_0 - (N_0 - I_0)e^{-\frac{\beta}{\gamma}(R(t)/N_0)}$ | $I(0)e^{(\beta-\gamma)t}$ |
| Nelder (84) | $qI(1 - (I/I^\infty)^\phi)$ | $I^\infty(1 + e^{-\phi(c+qt)})^{-1/\phi}$ | $I(0)e^{qt}$ |
| Flexible logistic (85) | $q[(1 + kt)^{1/k}]^{\mu-k}I(1 - I/I^\infty)$ | $I^\infty(1 + e^{-[c+qt(\mu,k)]})^{-1}$ | $I(0)e^{qt}$ |

Table S1: **Early growth patterns of selected models** 1) For all models, $I(0)$ represents the initial impact of a given product. 2) For the SIR model, to avoid duplicate usage of letter, we use A to represent number of current infected people. S corresponds to the number of potential users and R measures number of recovered people. The parameters satisfy the condition $S + A + R = N_0$. The impact of the product is captured by $I \equiv A + R$. 3) For the Flexible Logistic Growth model, μ and k are constants and $t(\mu, k) = [(1 + kt)^{\mu/k} - 1]/\mu$ for $\mu \neq 0$, $k \neq 0$, $t(\mu, k) = (1/k)\log(1 + kt)$ for $\mu = 0$, $k \neq 0$, $t(\mu, k) = (e^{\mu t} - 1)/\mu$, $\mu \neq 0$, $k = 0$, $t(\mu, k) = t$ for $\mu = 0$, $k = 0$.

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